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TECHNICAL REPORT

DEVELOPMENT OF MODELS FOR DEFENSE SYSTEMS PLANNING

Prepared by

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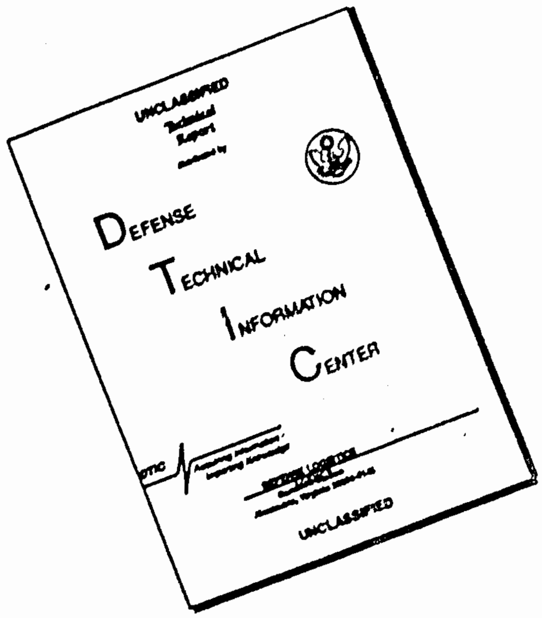

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SRL 2147 TR 70-2
Technical Report (U)

DEVELOPMENT OF MODELS FOR DEFENSE SYSTEMS PLANNING

Edited by
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SYSTEMS RESEARCH LABORATORY
Department of Industrial Engineering
The University of Michigan

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FOREWORD

This report describes research effort of the Systems Research Laboratory to develop analytical models of defense processes, principally the combat process. Part of the research was sponsored by the Office of Naval Research (ONR) under Contract No. N0014-67-A-0181-0012 and other parts by the Directorate, Weapon Systems Analysis, (DWSA) Office of the Assistant Vice Chief of Staff, U.S. Army, under Contract No. DAHC15-68-C-0314. Because of the intimate relationship between the research supported by these organizations, the results are combined in one document but issued under separate covers appropriate to the sponsoring agency. The report for the Directorate, Weapon Systems Analysis is entitled "Development of Analytical Models of Battalion Task Force Activities," Report Number SRL 1957 FR 70-1.

The report is comprised of a number of parts. Part A presents an overview of the differential models of combat developed in the research program and a summary of results for the reader who is interested in learning of the modeling approach without involvement in mathematical details. Parts B through F contain the mathematical developments. Part B presents the concepts, development details, and resultant models for the "attrition rate"--the principal element of the differential combat models. Parts C and D describe solution procedures and analysis results for homogeneous-force and heterogeneous-force battle models, respectively. The results of a small effort to analytically

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model reconnaissance activities are described in Part E. Part F presents research results for miscellaneous areas which are tangentially related to the main thread of research or, due to limited effort, only state the research approach.

The research program described in this report concerned only the development of generalized mathematical differential models of combat, rather than detailed models of specific combat situations. These general models have been applied to specific combat situations which had also been modeled by Monte-Carlo simulation methods. Comparisons between the differential models and a Monte-Carlo one showed that their predictions of combat results were essentially the same. This comparison activity was performed by Vector Research, Incorporated under contract DAHC15-70-C-0151 with the Directorate, Weapon Systems Analysis, after completion of the research reported herein. A short summary of the comparison results has been included in this report as an appendix to Part A to demonstrate that the differential models of combat, although abstract in form, can be usefully employed in defense planning activities.

Except for the Summary, Part A, each part of the report is comprised of chapters which are self-contained in so far as equation numbers, figures, etc. An attempt has been made to utilize consistent notation throughout the chapters using the definitions given in the list of symbols. Exceptions to this are either noted or self-evident in context of the particular

development. Frequent references are made to developments and equations among the various chapters and parts of the report to reduce redundancy of exposition. These references are made by the notation [capital letter, arabic numeral], where the capital letter identifies the part and the arabic numeral the chapter and section within the part.

The contents of this report represent the current views of the Systems Research Laboratory, Department of Industrial Engineering, The University of Michigan, and should not be considered as having official ONR, Department of the Navy, or DWSA, Department of the Army approval either expressed or implied until reviewed and evaluated by those agencies and subsequently endorsed.

We would like to acknowledge the contributions of Miss Mary Schnell, Mrs. Barbara MacAdam, Mrs. Pat Zangara, and Mrs. Bonnie Wood, who patiently typed and proofread the text of the report.

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SYMBOLS

This listing contains principal notation used in the report. Some symbols are used more than once; however, their meaning should be clear in context of a specific chapter or part of the report. Subscript notation has been omitted.

English Symbols

A	Blue attrition coefficient
A	Blue attrition-rate matrix
A	Total area searched
A	Total area searched by surveillance patrol
a_i	Area of i^{th} subarea searched
B	Red attrition coefficient
B	Red attrition-rate matrix
b_m [b_n]	Firing rate common to all units of the Blue [Red] force
C	A combined attrition-rate matrix
C	Terminal surface in E^+
c	A constant ratio of the Red to Blue attrition-rate functions
d	The difference m - n
d_i	Distance between subareas (i - 1) and i
d_o	The difference m - n at r = 0
E	Expected value operator
E	Blue allocation matrix
E^*	Optimal allocation strategy matrix for Blue force

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$E_M(t)$ [$E_N(t)$]	Total ammunition expenditure of a Blue [Red] unit up to time t in an engagement
E^+	Euclidean (I + J) space
e	Blue allocation factor
F_J	Average fraction of time that the J-type weapons are not advancing
$\hat{F}_v$	Corrected approximate expected fraction of damage to an area target in v volleys
$f_A(t)$ [$f_B(t)$]	Probability density function of the time between A's [B's] rounds
$\bar{f}_v$	Expected fraction of damage to an area target in v volleys
$\hat{f}_v$	Approximate expected fraction of damage to an area target in v volleys
g(t)	Probability density function for τ_d
H	Red allocation matrix
H*	Optimal allocation strategy matrix for Red
H_H	Probability that a hit after a hit destroys the target
H_M	Probability that a hit after a miss destroys the target
H_1	Probability that a hit on the first round destroys the target
h	Red allocation factor
I	Blue intelligence factor
I	Maximum number of Blue force groups
J	Maximum number of Red force groups
J	Jordan normal form of a matrix
K	Red intelligence factor

K	Conditional probability of destroying the target, given it is hit by a projectile
K_α $[K_\beta]$	Slope of Blue [Red] linear attrition-rate functions
L_A $[L_B]$	Lifetime of A's [B's] firepower subsystem
L_A^* $[L_B^*]$	Time A [B] detects his failure
ℓ	Number of target postures
$\ell(\tau)$	Probability density function for τ_d^*
M	Initial number of Blue forces
M_H	Probability that a miss after a hit destroys the target
M_M	Probability that a miss after a miss destroys the target
M_s	Number of surviving Blue units at the split range in the fire-support engagement
M_1	Probability that a miss on the first round destroys the target
M_2	Number of units in the Blue fire-support force
$M(t)$	Renewal function
ΔM_I	Number of Blue I-group losses in time increment ΔT
m	Number of Blue forces as a function of time or range
m_1	Number of units in the Blue moving forces in the fire-support engagement
N	Initial number of Red forces
N	Number of rounds fired to destroy a target
ΔN_J	Number of Red J-group losses in time increment ΔT
n	Number of Red forces as a function of time or range
n	Number of subareas searched by surveillance patrol
n_1	Number of rounds fired to get the first hit
n_2	Number of rounds to get $(z - 1)$ additional hits

P_K	Conditional probability of destroying the target given it is hit by a projectile
$P_{L,Q}$	Probability of acquiring a live target and terminating attention to that target before it is destroyed
$P(x)$	Payoff when the battle terminates at x on C
p	Rehitting probability
p	Conditional probability of a hit given the preceding round fired missed the target
p	Expected number of rounds required to destroy a target (E[N])
p	Percent force split in the fire support engagement
p_A [p_B]	A's [B's] single-shot kill probability
p_D	Probability of firing on a dead target
p_i	Probability of detecting a target in i^{th} subarea
p_L	Probability of firing on a live target
p_v	Probability that the target and observer are intervisible
p_V	Probability of firing in a void area
$p(t)$	Probability a round fired at time t destroys the target
P_1	First round hit probability
Q_M [Q_N]	Total ammunition requirements for the Blue [Red] force
Q_M^* [Q_N^*]	Adequate ammunition requirements for the Blue [Red] force
q_m [q_n]	Initial ammunition supplies for each Blue [Red] unit
q_m^* [q_n^*]	Sufficient ammunition supplies for each Blue [Red] unit

R_α $[R_\beta]$	Range at which a Blue [Red] weapon system first achieves a nonzero attrition rate
R_e	Range at which a weapon system (Blue and Red) first obtains a nonzero attrition rate (i.e., $R_e = R_\alpha = R_\beta$)
R_p	Radius of damage pattern
R_s	Range at which the Blue force splits in the fire-support engagement
R_t	Radius of target area
R_o	Range at which the battle begins
r	Range between forces (force separation)
$r_A(t)$ $[r_B(t)]$	Probability density function of A's [B's] lifetime
S_1	Probability of covering the target in one volley
s_n $[s_m]$	Distance of the Red [Blue] forces from some common reference
T	Time for a single Blue [Red] system to destroy a passive Red [Blue] target
T	Total time that the target is in the visible state
T	Duration of the engagement
T_A $[T_B]$	Time for A [B] to destroy a passive target, given he is free from failures
$\overline{T}_D$	The expected time to fire on a dead target before beginning search for another target
$\overline{T}_L$	The expected or average time to fire on a live target before beginning search for another target [same as E(T)]
$\overline{T}_{L,K}$	Mean time between the commencement of searches when a live target is acquired and destroyed by the acquiring unit
$\overline{T}_{L,Q}$	Mean time between the commencement of searches when live target is acquired but not killed by the acquiring unit

$\overline{T}_V$	The expected time to fire on a void area before beginning search for a target
t	Time variable
t	Time since the beginning of battle
U	Value of the payoff when optimal strategies are employed
u	Conditional probability of a hit given the preceding round fired hit the target
u_A [u_B]	Probability A's [B's] round fails
V	Speed of the main force
v	Relative speed between the Blue and Red forces $v_m - v_n$
v	Conditional probability of a hit following a miss but preceding the first hit
v	Speed of the surveillance patrol which advances to search area A
v	Speed of movement between subareas in surveillance activity
v_n [v_m]	Speed of Red [Blue] force
x	Damage pattern center of impact in the x direction
y	Damage pattern center of impact in the y direction
z	Number of hits required to destroy the target

Greek Symbols

α	Blue attrition rate
α_k	Probability A fails on round $k + 1$
α_o	Value of the Blue attrition rate at $r = 0$
$\alpha(r)$	Blue attrition-rate function
$\alpha(0)$	Value of the Blue attrition rate at $t = 0$

β	Red attrition rate
β_j	Probability B fails on round j + 1
β_o	Value of the Red attrition rate at r = 0
$\beta(r)$	Red attrition-rate function
$\beta(0)$	Value of the Red attrition rate at t = 0
$\gamma\Delta t$	Probability one round is fired in $(t, t + \Delta t)$
λ	Probability of destroying the target given a coverage
$\Pi_1(t)$	Probability that a target is visible at t
$\Pi_2(t)$	Probability that a target is not visible at t
ρ	The ratio m/n
ρ_o	The ratio m/n at r = 0
τ_a	Time to acquire targets
τ_b	Average time between rounds during the burst firing mode
τ_d	Time required to detect a target when it is continuously visible to the sensor
τ_d^*	Time to detect a target, given it is detected
τ_f	Projectile flight time
τ_h	Time to fire a round given the preceding round was a hit
τ_h	Time to fire the first round in the burst process after obtaining the first hit in the single-shot process
τ_m	Time to fire a round given the preceding round was a miss
τ_s	Time spent in the subarea if a target is not detected
τ_v	Time that the target remains visible
τ_1	Time to fire the first round

$\phi(t)$	Approximate expected fraction of damage to an area target in v volleys at time t
$\phi_c(t)$	Corrected approximate expected fraction of damage to an area target in v volleys at time t
ω	relative acceleration between the Blue and Red forces

PART A

OVERVIEW AND SUMMARY OF THE RESEARCH PROGRAM

Chapter 1

INTRODUCTION

Seth Bonder

The importance of employing quantitative approaches to military planning activities is well recognized.[1] Central to many of these activities, and of particular importance to weapon system planning studies (selection, tactical doctrine, etc.), is the requirement for methods to predict the effectiveness of combat units equipped with different mixes of weapon systems. It is further incumbent that the effectiveness estimating methods be related to decision variables under control of the military planner in a way such that the effect of their variation may be readily observed.[2]

The development of methods to measure or predict effectiveness of combat units, and identification of the variables which significantly contribute to combat effectiveness, has been limited for a number of reasons. By definition, measures of a combat unit's effectiveness should reflect the degree to which the unit accomplishes its mission. Additionally, it is well known that mission accomplishment is highly dependent upon the complex

[1]See Bonder (1970), Hitch and McKean (1960), and Enke (1967).

[2]These variables are often times referred to as conceptual combat functions, e.g., firepower, maneuver, intelligence, etc.

interaction of weapon system characteristics, threat variables, organization structures, tactics employed, and environmental conditions. One approach used has been to develop simple "indications" of combat effectiveness such as the "firepower score," "indices of combat effectiveness," and "single-shot kill probabilities." These indicators (a) do not measure accomplishment of unit missions, (b) essentially ignore most of the above factors which effect mission accomplishment, and (c) bear little relation to the physical combat process.

A second, and most heavily used, approach to predict effectiveness of combat units is that of Monte Carlo simulation. This approach is essentially one of modeling the combat situation in minute detail, explicitly including weapons system capabilities, threat, environment, and other factors which effect mission accomplishment. An example of the detail included is shown in Figure 1, which depicts a one-on-one duel, the basic combat activity in large-scale Monte Carlo simulations of ground combat. Random numbers are drawn to determine the time for each weapon to fire its first round. Focusing on the Blue weapon system, additional random numbers are drawn to determine the flight time of the first round to the target,[1] if the first round hit the target, and if the round destroyed the target. This process is simultaneously accomplished for the Red weapon system. If Blue has not destroyed Red with his first round, and

[1]This is usually treated as a range-dependent constant and need not be sampled by Monte Carlo methods.

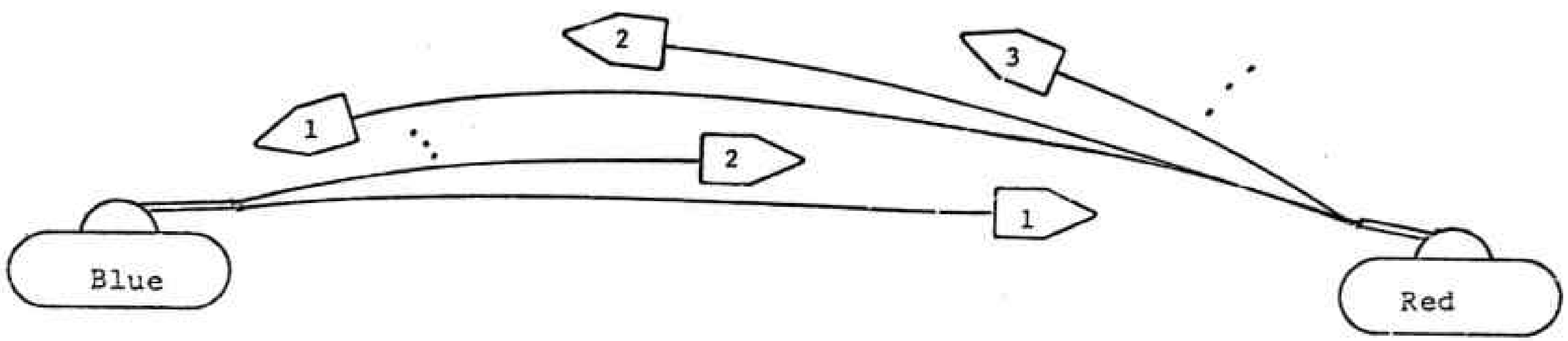

Figure 1 The Duel Process in a Monte Carlo Simulation

if he is alive himself, this process is repeated for Blue's second round, Red's second round, Blue's third round, and so on. The process is continued until one of the duelists is killed or the duel is terminated based on engagement rules built into the simulation.

These activities, and others, of every system are recorded during the course of the battle and eventually analyzed. Solution of such models is essentially an experiment in which the process is sampled and replicated a large number of times. The literature reflects the existence of a large number of Monte Carlo simulations used to analyze defense planning problems (Adams, 1961; Roberts, 1963; Quade, 1964; USACDC, 1969; Bishop and Clark, 1969).

Although Monte Carlo simulations are heavily employed in military planning circles, some meaningful drawbacks exist in their use as effectiveness assessment tools. Immediately evident is the loss in generality, since a new simulation must be developed for each class of weapon system or level of organization examined. Associated with a simulation is the large expenditure of time and financial resources for the development and utilization of the model. It would not be unreasonable to expect to spend 10 to 15 man-years in just developing a simulation of combat such as Carmonette (Adams, 1961) or Dyntacs (Bishop and Clark, 1969). Additionally, it would not be unreasonable to expect each replication of the simulation to require

10 to 20 minutes of computer time,[1] and anywhere from 10 to 60 replications for statistical stability of the results. The large number of variables usually included in simulations makes it extremely difficult to run parametric studies with the model to perform sensitivity analysis over the simulation assumptions and input data. This is due to both the statistical experimental design problems and money constraints which prohibit the large number of replications needed to determine the distribution of outcomes. Finally, and perhaps most importantly, the large amount of detail contained in the simulation makes it difficult to use as a tool for analysis, i.e., single out those independent variables which significantly contribute to the combat effectiveness.

In contrast to the Monte Carlo simulation approach, a limited amount of effort has been devoted to developing and using analytic (mathematical) models to predict the effectiveness of combat units. In this approach the physical combat or other military situation is studied and decomposed into its basic elements, mathematical descriptions of these elements are developed, and these element descriptions are integrated in an assumed overall mathematical structure of the process dynamics. Solutions are obtained by consistent mathematical operations giving rise to relationships between independent

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[1] Test runs with the Carmonette simulation required 2 minutes of computer time to simulate 1 minute of battle in a single replication (Adams, 1961, p. 35).

variables and the dependent ones of combat effectiveness. This approach has a number of obvious advantages both in its own right and as a powerful supplement to Monte Carlo simulations. Time and financial resources for development and utilization are usually markedly reduced. In analytic formulations, the relationship between indepandent factors of the process and the process output is usually explicitly presented, facilitating both sensitivity analysis and determination of those independent variables which significantly contribute to combat effectiveness. Finally, analytic structures are usually more general, thus facilitating more generalized use of the models across different combat organization levels and weapon systems.

Although analytic formulations appear to have a number of obvious advantages as military planning tools, only a limited number of them have been developed or employed as planning procedures. The most prominent of these are the Lanchester theories and the theory of stochastic duels, both of which are well documented in the literature (Dolanský, 1964; Ancker, 1967). The structure of initial Lanchester theories is given in [C, 1.0] and a summary of the stochastic duel literature is contained in [F, 1.1]. A brief summary of problems associated with their use as planning tools is given below.

The Lanchester theories of combat provide the means of describing combat between organizations comprised of numbers of heterogeneous weapons systems; however, general solutions

for the heterogeneous-force case do not exist. Excluding the apparent contradiction of results from verification studies (Engel, 1954; Willard, 1962), a number of other important deficiencies currently exist which preclude their use as planning tools. No means are available for predicting the *attrition coefficients*--a principal effectiveness input to the theory--as a function of the capabilities of the weapon systems. The mobility of weapon systems (an important aspect of their tactical use) is not explicitly considered, nor is the fact that the attrition coefficients vary when either or both combatants use mobile weapon systems, i.e., variations in force separation affect a weapon system's acquisition, firepower, and protection capabilities.

The relatively new theory of stochastic duels attempts to overcome a major deficiency of the Lanchester formulations--that of aggregating the weapon system parameters. Stochastic duel descriptions include basic weapon capabilities such as their firing times, hit probabilities, and kill probabilities. To date, this approach has been only partially successful. Although there has been an attempt to consider fundamental characteristics of weapons systems, the duels ignore some important parameters and place rather restrictive assumptions on the parameters. Application of the stochastic duel approa to multiple duels and, more importantly, large-scale battles requires increasingly more restrictive assumptions regarding the parameters and employment doctrine. As with the Lanches

approach, the stochastic duel descriptions virtually omit the effect of mobility on the outcome of engagements and the fact that the weapon parameters are time dependent when either or both combatants employ tactical mobility.

In summary, methods are needed to predict the effectiveness of combat units equipped with mixes of weapon systems. There is a heavy reliance on Monte Carlo simulations of combat for this purpose; however, there exists a number of significant deficiencies in their development and sole utilization as planning tools. Although analytic approaches appear to have some obvious advantages in their own right and as supplements to Monte Carlo simulations, deficiencies in the existing Lanchester and stochastic duel theories are sufficient to limit their use as planning tools.

The objective of the research program described herein is to develop analytic representations of combat and other military activities that can be used efficiently and effectively for planning purposes. The remainder of this part of the report presents an overview of the approach taken, a qualitative summary of the results obtained, and a brief description of additional research requirements. Parts B through F of the report contain the quantitative results, detailed mathematical developments, and solution procedures.

Chapter 2

AN ANALYTIC STRUCTURE OF COMBAT

Seth Bonder and Robert Farrell

In a broad sense the primary objective of our research is the development of analytic structures that can be used to predict the *results* of an artificial history of combat. Essentially, this would be a trajectory or trace of time, geometry, casualties, and resources expended for both forces.[1] Measures of combat effectiveness such as the ratio of surviving forces at the objective, time to overrun the objective, and the amount of terrain controlled are then determined from these results of battle.

Ideally, there exists some functional relationship between the results of battle and the initial numbers of forces, types and capabilities of the weapons systems, the doctrine of employment, and the environment. Thus, we would like to specify the function f shown below.

$$\begin{pmatrix}\text{Results}\\ \text{of}\\ \text{Battle}\end{pmatrix} = f \left\{ \begin{array}{l} \text{Numbers of Forces} \\ \text{Types of Weapon Systems} \\ \text{Weapon Capabilities} \\ \text{Doctrine of Employment} \\ \quad \text{(tactics, organization)} \\ \text{Environment} \end{array} \right.$$

[1] It is important to recognize that what is being developed is a descriptive theory of combat activities and not a normative one which specifies an optimum force structure, although some optimization methods have been examined.

Unfortunately, it is not known how to hypothesize such a function directly, nor is there sufficient data to develop it empirically. Because of this, we attempt to approximate what happens in a small period of time during the battle. That is, for each side, it is hypothesized that in a short period of time

(a) locations change due to tactical movement,

(b) weapon systems are attrited by enemy activity,

(c) resources are expended, and

(d) personnel become casualties due to enemy activity.[1]

Focusing on the loss of weapon systems and personnel, it is assumed that, if the state of the battle at the beginning of the small interval is known, and the activity that takes place during the interval is known, the *rate* at which weapons systems and personnel are attrited during this small interval can be predicted.[2] It is because of this rate focus that the mathematical structure employed to model the combat activity is that of differential equations.

For convenience, names are assigned to the numbers of different groups of systems in each force. Let

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[1] Reserve commitment and resupply during the small interval of time are also possible but are omitted for presentation purposes.

[2] This essentially is the concept of measurable attrition rates formulated by F. W. Lanchester (1916).

m_i = the number of surviving Blue units of the i^{th} group ($i = 1,2,\ldots,I$).

n_j = the number of surviving Red units of the j^{th} group ($j = 1,2,\ldots,J$).

Different groups are determined by their ability to attrit weapons systems of an opposing group. Therefore, missile weapon systems and rapid-fire machine guns form different groups since the rate at which they can attrit targets of an opposing group are different. Additionally, similar weapon system types can form different groups if they are at different ranges to the target and this range difference affects their ability to attrit it. Thus, a tank platoon 1,000 meters from the target is a different group than another tank platoon 3,000 meters from the target.

The overall analytic structure of the combat activity is based on assumptions that

(a) the rate of loss of units in the j^{th} Red group due to the i^{th} Blue group is proportional to the number of units in the i^{th} Blue group with a proportionality factor called the *attrition coefficient*, and

(b) the rate of loss of units in the j^{th} Red group in total is the sum of the rates of losses due to different i^{th} Blue groups.

Mathematically, these assumptions take the form of the following coupled sets of differential equations:[1,2]

$$\frac{dn_j}{dt} = -\sum_{i=1}^{I} A_{ij}(r)m_i \qquad \text{for } j = 1,2,\ldots,J \qquad [1]$$

$$\frac{dm_i}{dt} = -\sum_{j=1}^{J} B_{ji}(r)n_j \qquad \text{for } i = 1,2,\ldots,I\ , \qquad [2]$$

where

$A_{ij}(r)$ = the utilized per system effectiveness of systems in the i^{th} Blue group against the j^{th} Red target at range r. This is called the Blue attrition coefficient.

$B_{ji}(r)$ = the utilized per system effectiveness of systems in the j^{th} Red group against the i^{th} Blue target at range r. This is called the Red attrition coefficient.

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1 Although the variable r is used to designate the range between the firing weapon group and the target group, it should be noted that, in application of the model, actual time trajectories and positions of each group can be considered.

2 Although not explicitly shown, resources expended are explicitly contained in the development of the A_{ij} [see (B, 2.0)] and can be determined directly from the model, as noted in [F, 3.0].

It is noted that this formulation is a deterministic one which treats the numbers of surviving forces (m_i and n_j) as continuous variables, while clearly the actual battle activity is a random phenomenon and m_i and n_j are integer-valued variables. Although many probabilistic arguments are contained in this formulation (as shown in Parts B through F of this report), the output of the model is a deterministic trajectory of the surviving numbers of forces. The reasons for this deterministic formulation, instead of a stochastic one of the same process, are given in [B, 1.0]. It is of interest to note that research done on comparing the deterministic and stochastic formulations for the homogeneous-force case (only one force group on each side) indicates that the deterministic formulations are reasonably good approximations of the expected number of survivors if there is a small probability that either side is annihilated. Additionally, in many defense studies that employ Monte Carlo simulations, typically only the expected results are considered in the decision-making process.

The attrition coefficients (A_{ij} and B_{ji}) are, as one would expect, complex functions of the weapon capabilities, target characteristics, distribution of the targets, allocation procedures for assigning weapons to targets, etc. The model attempts to reflect these complexities by partitioning the total attrition process into four distinct ones:

1. The effectiveness of weapons sytems while firing on live targets,
2. The allocation procedure of assigning weapons to targets,
3. The inefficiency of fire when other than live targets are engaged, and
4. The effect of terrain on limiting the firing activity and on mobility of the systems.

The latter was not examined in the research program; however, a means of incorporating these effects was included in the comparison of the model predictions with that of a Monte Carlo simulation model, as described in Appendix A.

The first three effects are included in the attrition coefficient as

$$A_{ij}(r) = \alpha_{ij}(r)e_{ij}(r)I_{ij}(r) \tag{3}$$

$$B_{ji}(r) = \beta_{ji}(r)h_{ji}(r)K_{ji}(r) \ , \tag{4}$$

where

$\alpha_{ij}(r)$ = *the attrition rate*--the rate at which an individual system in the i^{th} Blue group destroys live j^{th} group Red targets at range r when it is firing at them,

$e_{ij}(r)$ = *the allocation factor*--the proportion of the i^{th} Blue group systems assigned to fire on the j^{th} group Red targets which are at range r,

$I_{ij}(r)$ = *the intelligence factor*--the proportion of the i^{th} group firing Blue weapons allocated to the j^{th} Red group which are actually engaging live j^{th} group Red targets at range r.

Similar definitions exist for the components of the Red attrition coefficient, B_{ji}.

Major emphasis in the research prog.am has been on the development of methods for predicting these inputs and the development of solutions of the resultant coupled sets of differential equations. The methods developed to date and results of the solution procedures are summarized in Chapters 3 and 4 of this part of the report. Chapter 5 briefly describes results of related modeling of reconnaissance activities and an extension of the stochastic duel models of combat. Areas for future research are also noted in Chapter 5.

Chapter 3

ATTRITION COEFFICIENT PREDICTION METHODS

Seth Bonder and Robert Farrell

As shown in the previous chapter, the attrition coefficient is made up of the attrition rate, the allocation factor, and the intelligence factor. Research has been devoted to the development of methods to predict these inputs with major emphasis on prediction of the attrition rate. Detailed descriptions of attrition-rate prediction methods are given in Part B of the report. Allocation factor research is described in [D, 2.0] and formulae for predicting the intelligence factor are developed in [E, 1.0].

3.1 The Attrition Rate

Basic to the differential model or theory of combat is the attrition rate, which is the rate at which a weapon system can destroy live targets when it is firing at them. In the classical Lanchester theories, the attrition rate has been assumed constant or state-dependent (dependent on the numbers of surviving Red and Blue forces). The ability to obtain, other than by hindsight, a satisfactory estimate of the attrition rate for future engagements has limited the use of classical Lanchester theories for planning.

The concept of the attrition rate formulated in this research program is described in [B, 1.0]. Simply, it is

assumed to be dependent on a multitude of physical parameters of a weapon system which describe its capabilities in such areas as acquisition, firing accuracy, delivery rate, and warhead lethality. This dependency gives rise to two distinct variations in the attrition rate--variation with range to the target and chance variation at any specific range.[1] A mathematical structure of heterogeneous-force combat which includes the range and chance variations explicitly cannot be analytically solved with existing mathematical techniques. For this reason we have suppressed the explicit chance variation and used average attrition rates. This leads directly to the combat formulation given by equations 1 and 2 (see page 14). In this formulation we can consider the range variation of the attrition rate explicitly and somewhat independently of the chance variation at each specific range to the target.

Based on some logical and mathematical arguments, it has been shown that the appropriate average value definition of the attrition rate to use (for a specific range) with equations 1 and 2 is

$$\alpha_{ij}(\text{at range } r) \overset{\text{def.}}{=\!=} \frac{1}{E[T_{ij}|r]}\,, \qquad [5]$$

[1]For clarity of discussion, variations in the attrition rate due to changes in target posture, environmental effect, etc., which can be included in the model, are not presented.

where

$E[T_{ij}|r]$ = the expected time for a single Blue system of the i^{th} group to destroy a passive j^{th} group Red target, given the target is at range r.

This definition for an average value of the attrition rate at range r is equivalent to the harmonic mean of the attrition rate when it is viewed as a random variable at range r. This definition also leads naturally to defining the range variation of the attrition rate as the variation in the reciprocal of $E[T_{ij}|r]$ as the range to the target changes. The range variation is called the *attrition-rate function* and is denoted by $\alpha_{ij}(r)$, as used in the differential equation structure of combat.

Based on the above discussions, research on attrition rates has been concerned primarily with the development of *time-to-kill* probability distributions and their expected values for a spectrum of weapon systems. The distribution for the time-to-kill random variable is developed by consideration of the number of rounds expended to achieve the kill. Thus, the amount of ammunition resources expended can be obtained directly for a specific combat activity. Essentially, what is done is to take the physical process of the duel (which is basic to Monte Carlo simulations) and model the dynamics of this process mathematically.

To ensure that the attrition rates developed are general, a taxonomy of weapons systems that is not dependent on physical hardware characteristics (such as caliber) was developed. Rather, the taxonomy reflects characteristics of weapons systems that would affect the methods used in predicting the attrition rates.

The taxonomy is shown in Figure 2. Weapon systems are first classified by their lethality characteristics as having either impact-to-kill mechanisms or area-lethality effects. Within each of these categories, we have found it useful to further classify weapon systems on the basis of their methods of using firing information to control the system aim point and their delivery characteristics, i.e., the firing doctrine employed.

Methods have been developed that allow the prediction of attrition rates for many of the weapon systems shown in the taxonomy. The first cases analyzed involved single-tube firings in which launch of a projectile occurred only after the observation of the effects of the preceding round. These are called "repeated single-shot" doctrines in our schema, and are sometimes called "shoot-look-shoot" doctrines by other analysts. Analyses have been undertaken of two subclasses: (a) those in which no use is made of information obtained from observations and (b) those in which the observations are treated distinctly depending on whether they are a hit or a miss, leading to different types of correction in aim point for these two cases.

LETHALITY MECHANISM:

1. IMPACT
2. AREA

FIRE DOCTRINE:

1. REPEATED SINGLE SHOT:
   - *A) WITHOUT FEEDBACK CONTROL OF AIM POINT
   - *B) WITH FEEDBACK ON IMMEDIATELY PRECEDING ROUND (MARKOV FIRE)
   - C) WITH COMPLEX FEEDBACK
2. BURST FIRE:
   - *A) WITHOUT AIM CHANGE OR DRIFT IN OR BETWEEN BURSTS
   - *B) WITH AIM DRIFT IN BURSTS, AIM REFIXED TO ORIGINAL AIM POINT FOR EACH BURST
   - C) WITH AIM DRIFT, RE-AIM BETWEEN BURSTS
3. MULTIPLE-TUBE FIRING: FEEDBACK SITUATIONS (1A, B, C)
   - *A) SALVO OR VOLLEY
4. MIXED-MODE FIRING:
   - A) ADJUSTMENT FOLLOWED BY MULTIPLE-TUBE FIRE
   - *B) ADJUSTMENT FOLLOWED BY BURST FIRE

* INDICATES THAT ANALYSIS OF THIS CATEGORY HAS BEEN PERFORMED.

Figure 2 Weapon System Classification for the Development of Attrition Rates

This subclass is called "Markov fire." A completely general time-to-kill probability distribution for Markov fire systems has been developed. Weapon system parameters that are included explicitly in the distribution are shown in Figure 3. Methods of predicting these parameters from basic hardware considerations are well known.

The more complex doctrines involving "multiple-tube firings" and "burst fire," have been analyzed separately. These are classes of systems for which the projectiles may be launched before observation of previous round effects. Burst-fire cases analyzed include those in which rounds are all identical with respect to accuracy (no drifting or controlled alteration of the aim point) and those in which the rounds within a burst vary, but the bursts are resighted to the same aim point. All present analyses have been based on fixed-length bursts. The complex case in which bursts are re-aimed on the basis of observation has not been analyzed.

Preliminary analyses have been conducted of multiple-tube firing cases, and it has been determined that the attrition rate for both volley and salvo fire may be represented by the same formulae. The method developed considers a weapon system which, perhaps not knowing the exact location of targets, fires indirectly into an area with a projectile that delivers damage-producing effects over part of the area. Parameters included in the method are shown in Figure 4. Each of these parameters can be predicted from basic hardware characteristics of weapons systems and targets.

TIME TO ACQUIRE A TARGET

TIME TO FIRE THE FIRST ROUND

TIME TO FIRE A ROUND FOLLOWING A HIT

TIME TO FIRE A ROUND FOLLOWING A MISS

PROJECTILE FLIGHT TIME

PROBABILITY OF A HIT ON FIRST ROUND

PROBABILITY OF A HIT ON A ROUND FOLLOWING A HIT

PROBABILITY OF A HIT ON A ROUND FOLLOWING A MISS

PROBABILITY OF DESTROYING A TARGET GIVEN IT IS HIT

PROBABILITY OF DESTROYING A TARGET GIVEN IT IS MISSED

Figure 3 Factors Included in Attrition Rate for Single-Shot Markov-Fire Weapon Systems

WEAPON AIMING AND BALLISTIC ERRORS

TARGET LOCATION ERRORS

WEAPON FIRING RATE

VOLLEY DAMAGE-PATTERN RADIUS

TARGET DISTRIBUTION

TARGET RADIUS

TARGET POSTURE

PROBABILITY THAT THE TARGET IS DESTROYED GIVEN IT IS COVERED BY DAMAGE PATTERN

Figure 4 Factors Considered in Attrition Rate for Indirect, Area-Fire Weapons

Finally the mixed mode firing doctrine in which a period of single-shot fire is followed by burst fire has also been analyzed.

3.2 *The Allocation Factor*

As noted earlier, the allocation factor is the proportion of the i^{th} Blue group systems assigned to fire on j^{th} group Red targets. This is included since only those systems directing their fire (or other lethal effects) on the j^{th} group or its area are likely to cause attrition of the target. The allocation factor may be input by military judgment reflecting the assignment strategies deemed most appropriate to the tactical situation. This factor may be input directly or determined from a priority or target worth scheme.

Research in this area has focused on the determination of optimal or good allocation strategies when the battle dynamics are described by the coupled sets of heterogeneous differential equations shown earlier. The research is described in [D, 2.0]. The results obtained are based on the following assumptions:

(1) Zero time is required to switch from one target group to another,

(2) Projectile flight times are small, and

(3) The groups have perfect control and intelligence.

The research has shown that, for linear payoff functions, it is ineffective for individual weapon types to distribute their fire over different target groups. That is, all i-group weapons should engage all j-group targets with no splitting of fire allocation within a group. The optimal assignment strategies are such that all weapons of a single group should be assigned to a single group in the opponent's arsenal. Mathematically,

$$e_{ij}(r) = \begin{cases} 1 & \text{for } j = K \\ 0 & \text{for } j \neq K \end{cases} \qquad \text{for } i = 1,2,\ldots,I \qquad [6]$$

$$h_{ji}(r) = \begin{cases} 1 & \text{for } i = L \\ 0 & \text{for } i \neq L \end{cases} \qquad \text{for } j = 1,2,\ldots,J \ , \qquad [7]$$

where K and L denote a specific weapon type in the Red and Blue forces, respectively.

The research has also shown that the choice of group to be fired upon is independent of the number of weapons in the firing or target group. The class to be fired upon is selected by determining the maximum attrition rates on the marginal utilities of the opposing groups and not

directly by the number of weapons in the opposing groups.[1] Furthermore, although previous research (Snow, 1948) employed the assumption that the allocation coefficients were constant throughout the battle, it has been shown that switching surfaces do exist, i.e., the optimal allocation strategy changes during the battle even though none of the Blue or Red force groups are annihilated.

Closed-form analytic solutions for the optimal allocation strategies (initial allocation and switching surfaces) have been obtained for the two-on-one battle, i.e., two groups on one side and one on the other. The method used is applicable to higher-order battles; however, the mathematics gets extremely cumbersome.

3.3 The Intelligence Factor

As previously noted, the intelligence factor is the proportion of the i^{th} group firing Blue weapons allocated to the j^{th} Red group which are actually engaging live j^{th} group Red targets. This factor is included to consider the loss in efficiency (effectiveness) of a firing weapon when it is firing on either targets already attrited or on areas that

[1]This has an obvious implication on intelligence requirements during a battle for allocation. All that needs to be known is that there exists a live j-group target and not the number of live weapon systems in it.

are void of targets. Research in this area is described in [E, 1.0] and suggests that the intelligence factor should be predicted as

$$I(r) = \frac{p_L \bar{T}_L}{p_V \bar{T}_V + p_D \bar{T}_D + p_L \bar{T}_L}, \tag{8}$$

where

p_L = the probability of firing on a live target,

p_D = the probability of firing on a dead target,

p_V = the probability of firing in a void area,

$\bar{T}_L$ = the expected or average time to fire on a live target,

$\bar{T}_D$ = the expected or average time to fire on a dead target,

$\bar{T}_V$ = the expected time to fire on a void area.

At the present time, only the parameter $\bar{T}_L$, which is equal to the expected time to defeat a live target,[1] can be predicted as input. Research is required to develop methods to estimate the other parameters.

---

[1]That is, $\bar{T}_L$ is equivalent to what was previously referred to as the expected time to kill a target, $E[T_{ij}|r]$.

Chapter 4

COMBAT MODEL SOLUTION PROCEDURES AND RESULTS

Seth Bonder

The basic structure assumed to describe the combat activity was given by the coupled sets of differential equations

$$\frac{dn_j}{dt} = -\sum_{i=1}^{I} A_{ij}(r)m_i \qquad \text{for } j = 1,2,\ldots,J$$

$$\frac{dm_i}{dt} = -\sum_{j=1}^{J} B_{ji}(r)n_j \qquad \text{for } i = 1,2,\ldots,I.$$

The preceding chapter summarized methods that have been developed to predict inputs to these equations--the attrition rate, the allocation factor, and the intelligence factor. This chapter briefly presents results of research that has been directed to obtaining solutions for the above equations, where a solution is taken to be the trajectory of surviving forces of each type during the battle as a function of basic inputs and initial numbers of forces.[1]

Ideally, it would be desirable to have the solutions in simple, closed form which would readily portray the relationship between the independent factors of the combat process and the surviving numbers of forces. This would facilitate

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[1]Logistics and locations of survivors can also be determined as part of the solution, but are omitted in this discussion.

both sensitivity analysis and determination of those independent variables which significantly contribute to combat effectiveness. Attempts to obtain such closed-form solutions have focused on simplified cases of the combat equations in order to obtain some insight into the solution procedures and problems related thereto. These simplified cases include (a) homogeneous-force battles (one group on each side) and (b) constant-coefficient, heterogeneous-force battles. A summary of the results of these research efforts are presented in succeeding sections. Details of the homogeneous- and heterogeneous-force battle solutions are given in Parts C and D, respectively. A numerical solution procedure was developed to solve the equations for simplified tactical situations involving heterogeneous forces and variable attrition coefficients. This procedure is described in [D, 3.0].

4.1 Homogeneous-Force Results

We considered first the simplified case of homogeneous-force battles with unity intelligence coefficients.[1] The general heterogeneous equations noted above reduce to

$$\frac{dn(t)}{dt} = -\alpha(r)m(t) \tag{9}$$

$$\frac{dm(t)}{dt} = -\beta(r)n(t). \tag{10}$$

[1]All research presented in this report has considered unity intelligence coefficients.

Since there is only one group on each side, the allocation factor is also equal to unity for each force. In these equations explicit notation showing the time and range dependencies are given.

In order to include explicit consideration of some dimensions of mobility, the one-dimensional battlefield coordinate system shown in Figure 5 was considered. The symbols s_n and s_m, are the distances of the Red (n) and Blue (m) forces, respectively, from some common reference. The above equations can be converted to the space domain depicted in Figure 5, resulting in the following differential equations:

$$\frac{d^2 n}{dr^2} + \left[\frac{\omega}{v^2} - \frac{1}{\alpha}\frac{d\alpha}{dr}\right]\frac{dn}{dr} - \left(\frac{\alpha\beta}{v^2}\right) n = 0 \qquad (11)$$

$$\frac{d^2 m}{dr^2} + \left[\frac{\omega}{v^2} - \frac{1}{\beta}\frac{d\beta}{dr}\right]\frac{dm}{dr} - \left(\frac{\alpha\beta}{v^2}\right) m = 0 \;. \qquad (12)$$

These equations explicitly include maneuver characteristics of the forces such as speed (v) and acceleration (ω) and the range variation in attrition rates when the forces employ mobile weapon systems.

The solution of these equations required knowledge of the attrition-rate functions,[1] $\alpha(r)$ and $\beta(r)$ for the Blue and

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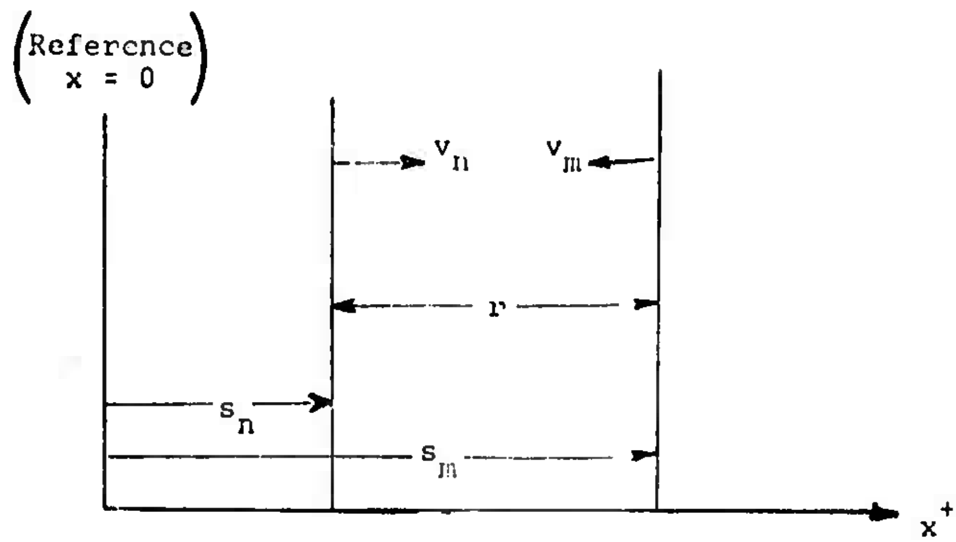
[1]It was noted in the preceding chapter that the attrition-rate function is defined to be the variation with range in the reciprocal of the expected time-to-destroy a target.



where

s_n (s_m) = the distances of the Red (Blue) forces from some common references.

r = force separation,

v_n (v_m) = velocity of the Red (Blue) force.

v = relative velocity between the Blue and Red force (v_m - v_n).

Figure 5 One-Dimensional Battlefield Coordinate System

Red weapons systems, respectively. Examination of data for some representative weapons systems suggested a number of forms for the attrition-rate functions, some of which are shown in Figure 6. These characteristic shapes were given appropriate mathematical descriptions, e.g., linear, quadratic, exponential, and cosine attrition-rate functions. In each case the range R_α is that force separation at which the weapon first attains a nonzero rate of attriting targets.

Attempts were made to obtain closed-form solutions for the homogeneous-force battle equations with these attrition-rate functions under the assumption that the acceleration of forces was zero ($\omega = 0$), i.e., a constant-speed battle. For example, assumptions of linear attrition-rate functions for both Red and Blue weapons are shown in Figure 7(a). Here R_α and R_β are the ranges at which the Blue and Red weapons systems, respectively, first achieve nonzero attrition rates. The resultant equations could not be solved in closed form without further assuming a constant *ratio* of Red to Blue attrition-rate functions. This last assumption for linear attrition-rate functions is shown in Figure 7(b). A general closed-form solution was developed for any pair of attrition-rate functions such that $\beta(r)/\alpha(r)$ = constant.

Even with these overly simplified, restrictive assumptions, solutions to the variable-coefficient differential equations gave rise to some interesting insights and

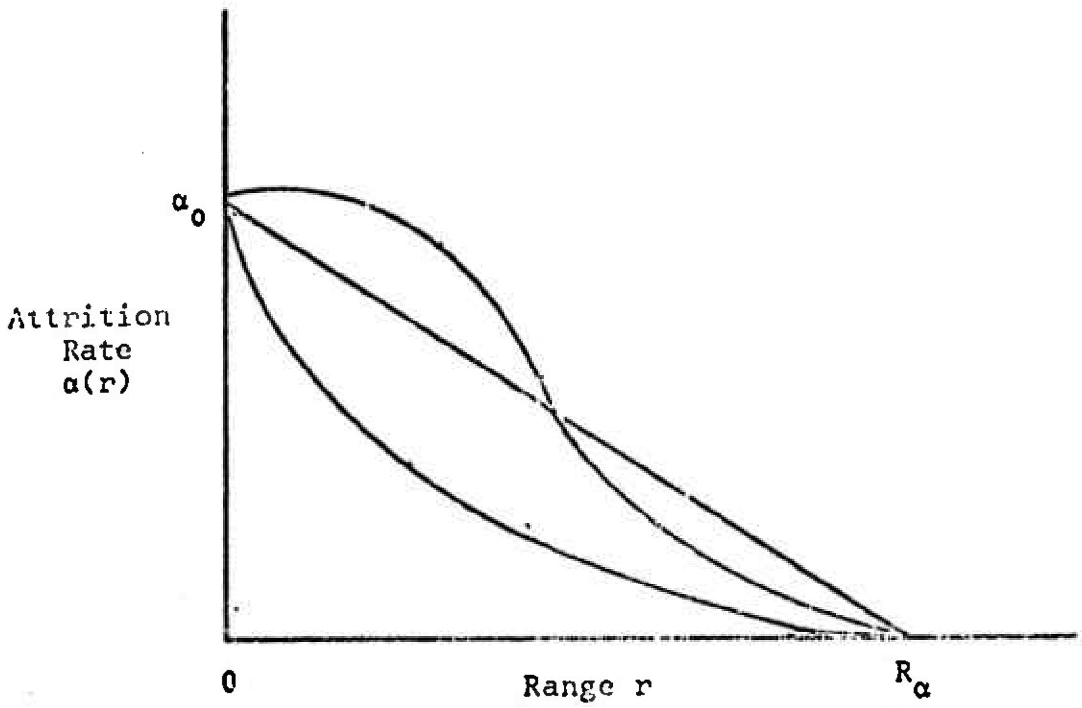


Figure 6 Attrition-Rate Variations with Range

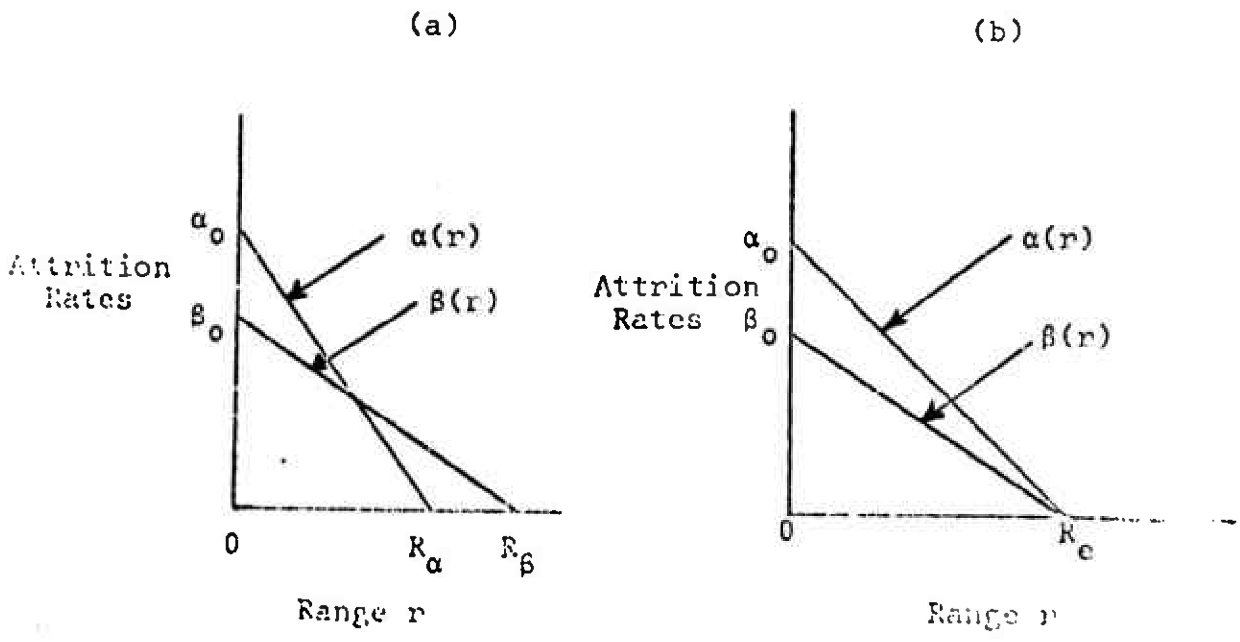


Figure 7 Attrition-Rate Assumptions

comparisons with existing theories. In particular, the classical constant-coefficient Lanchester formulation of this problem suggests that a Blue force will lose a battle when

$$\alpha M^2 < \beta N^2 ,$$

where M and N are the initial numbers of Blue and Red forces, respectively. This lose condition implies complete annihilation of the losing force.

Analysis of the variable-coefficient solutions, however, indicates that this win-or-lose condition is completely misleading. Rather, one should consider some measures of effectiveness (numbers of survivors, difference of survivors, ratio of survivors, etc.) at the end of the battle instead of the complete annihilation conditions. Thus, one may choose to consider any or all of the above measures of effectiveness when the force separation is zero (the attacker crosses over the defended line) or some prespecified breakpoint in terms of survivors and/or force separation. When this is done, then the results of the battle are highly dependent on the assault speed and the relationship between the *initial*, *linear*, and *quadratic* conditions defined below:

Initial Condition:

$$M \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} N$$

Linear Condition:

$$\alpha_0 M \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \beta_0 N$$

Quadratic Condition:

$$\alpha_0 M^2 \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \beta_0 N^2 ,$$

where α_0 and β_0 are the attrition rates for Blue and Red weapons, respectively, when their force separation is zero. The effect of these conditions and the use of mobility as measured by the assault speed are shown in Figures 8 through 11. The figures show the effect of the assault speed on the difference and ratio of surviving forces at the end of the battle.

The conditions shown in Figures 8 and 9 suggest, by classical Lanchester analysis, that the Blue force will be annihilated. This is true if their assault speed is less than 4 mph. However, increasing their assault speed to approximately 20 mph will result in their arriving at the defended position with a superiority of 14 units (where the initial superiority was 20) or a ratio of 2.9 to 1, where the initial ratio was 3 to 1. These figures are suggestive of two phenomena:

1. Attacking with sufficient speed is a means of conserving one's own force, i.e., get the enemy before he gets you. This we might term a saturation principle in that we saturate the enemy's retaliatory firepower capability with maneuver.

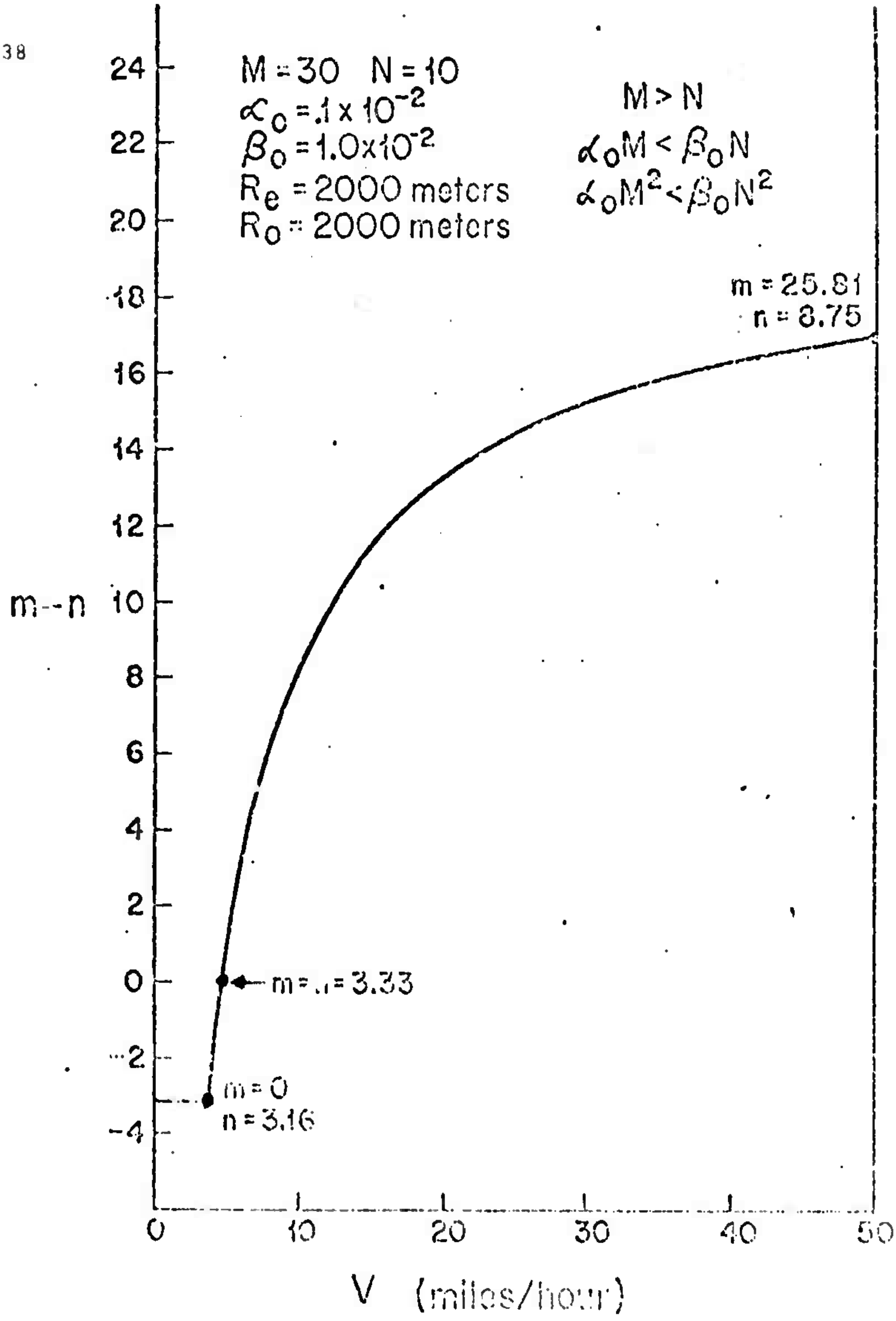


Figure 8 Force Difference at r = 0 (Constant-Ratio, Linear Attrition Rates)

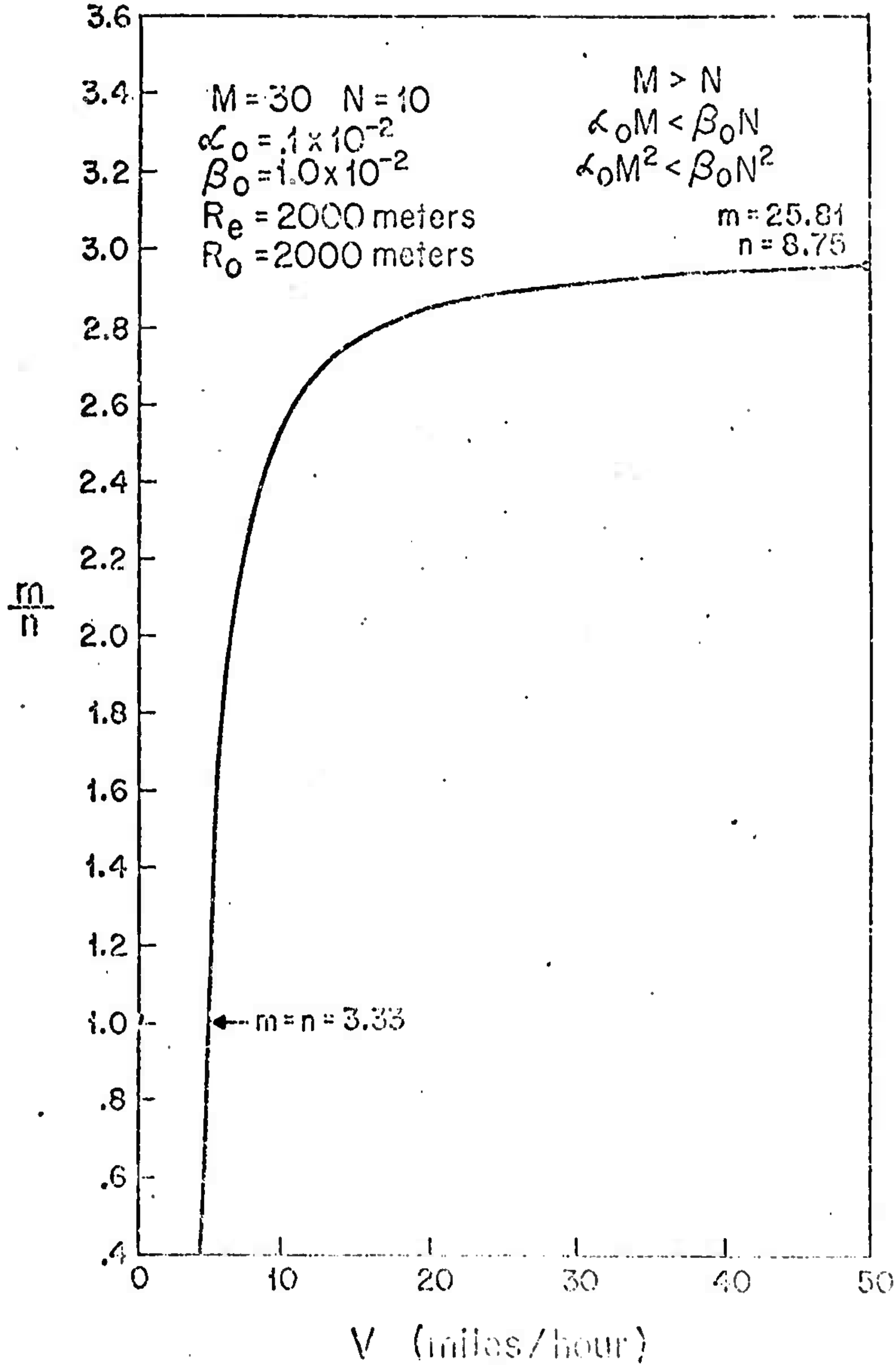


Figure 9 Force Ratio at r = 0 (Constant-Ratio, Linear Attrition Rates)

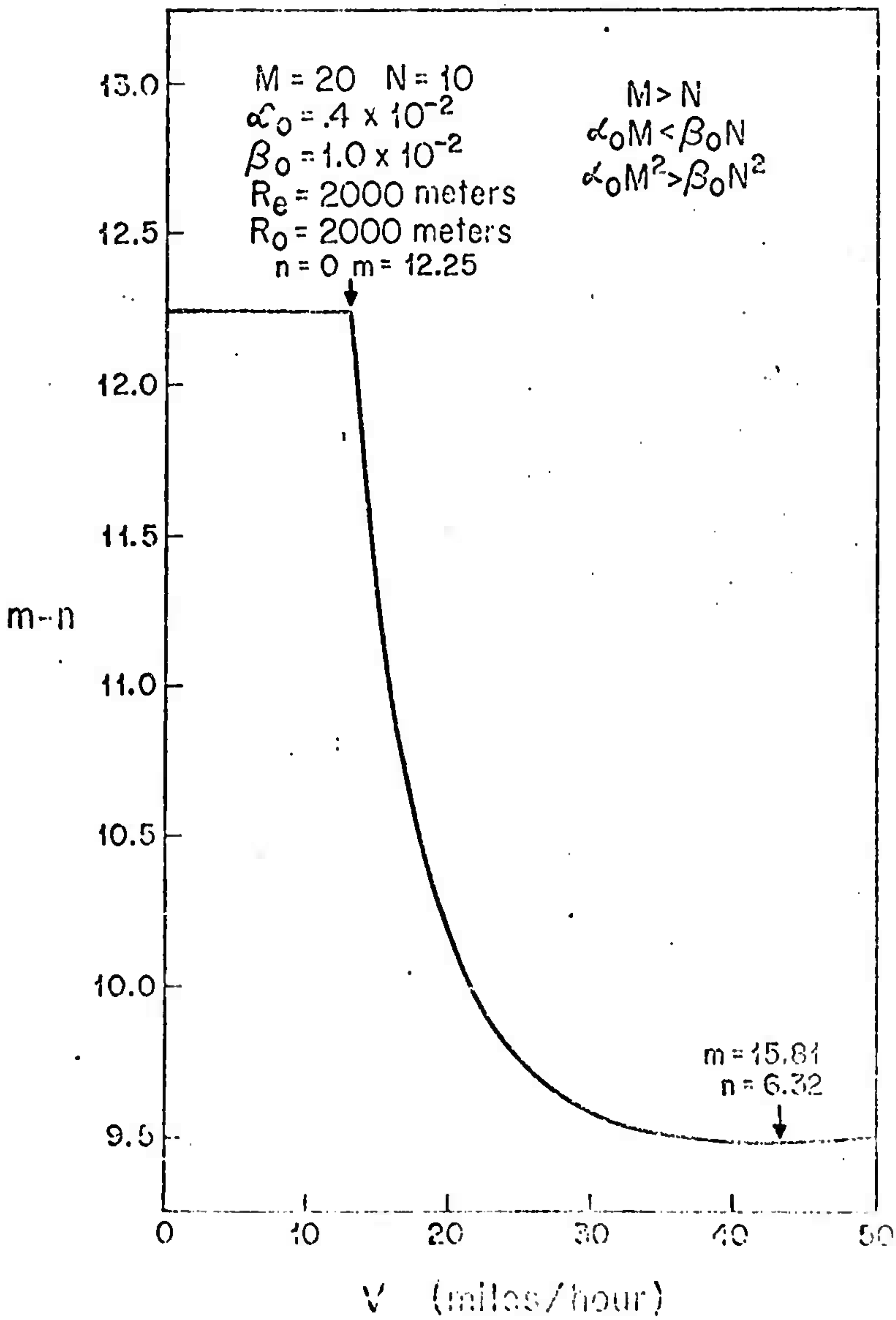


Figure 10 Force Difference at r = 0 (Constant-Ratio, Linear Attrition Rates)

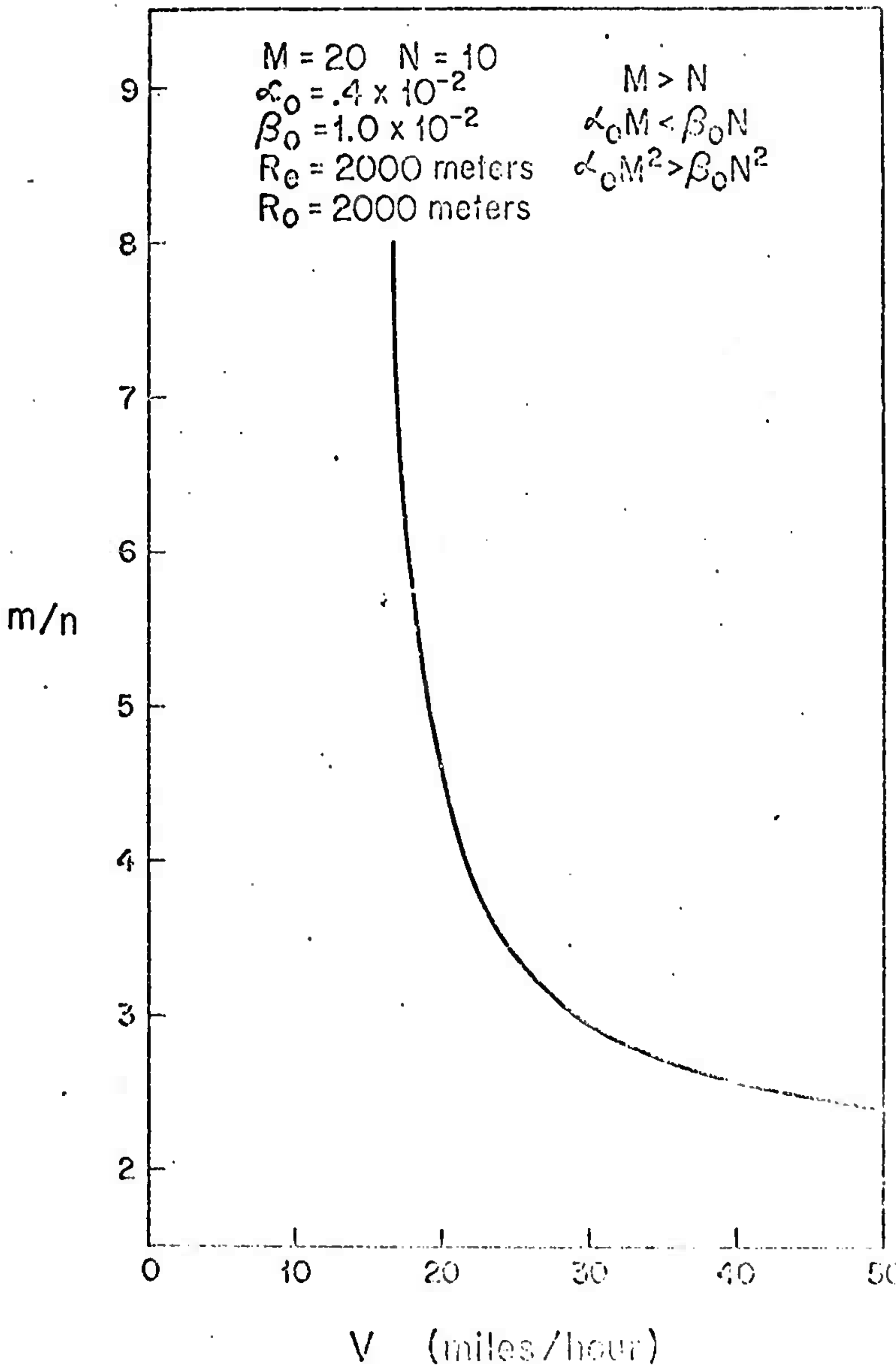


Figure 11 Force Ratio at r = 0 (Constant-Ratio, Linear Attrition Rates)

2. Increasing the assault speed increases the saturation effect; however, this effect has a decreasing marginal benefit.

The decreasing marginal utility of increasing assault speed is evidenced in both Figures 8 and 9; however, it is more pronounced in the ratio measure of effectiveness.

In contrast the these results, the conditions of Figures 10 and 11 suggest, by classical Lanchester analysis, that the Blue force will annihilate the Red force. This will occur only if the Blue force assault speed is less than 13 mph. Increasing their assault speed above this will result in their arriving at the objective with a lower superiority, measured by the difference and ratio of forces. It is interesting to note that when the measure of effectiveness is the force difference at the objective, there is a unique worst speed for the Blue force to attack; however, the ratio of surviving forces continues to decrease with increasing assault speed.

Although closed-form solutions to the homogeneous-force combat equations when the ratio $\beta(r)/\alpha(r)$ is not constant have not been obtained to date, research efforts have been directed to obtaining parity conditions (conditions leading to equal numbers of survivors on both sides at the end of the battle) Based on the work described above, we felt that

these conditions would depend not only on the force sizes[1] but also on the shape of the attrition-rate functions, the effective ranges, the range at which the battle is initiated, and the mobility of the attacking force.

Approximate solutions to the parity conditions have been obtained analytically (see [C, 4]); however, they have not provided a great deal of insight to date. Analog computer solutions to the equations, however, have tended to support the above conjectures. The analog computer provides a visual display of the solution space when parameters such as initial number of forces, assault speed, effective range of the weapons, opening range of the battle, etc., are varied. Systematic variations of these parameters were made to observe the trajectory of the parity conditions ($m = n$ at range $r = 0$). These are described in [C, 5].

Some typical plots of the solutions are shown in Figures 12, 13, and 14 for the absolute number of survivors, the difference in survivors, and the ratio of survivors, respectively, at the end of the battle. The parity points for variations in the initial numbers of the Red force are indicated by solid circles. Immediately obvious from these figures is the fact that the assault speed is an integral factor in predicting parity points. More importantly, there appear to be optimal assault speeds such that deviations from these optima can have significant effects on the battle results.

[1] The principal factors in the classical Lanchester parity conditions.

Figure 12 Number of Survivors at Range r = 0 (Linear Attrition Rate $R_\alpha \neq R_\beta$)

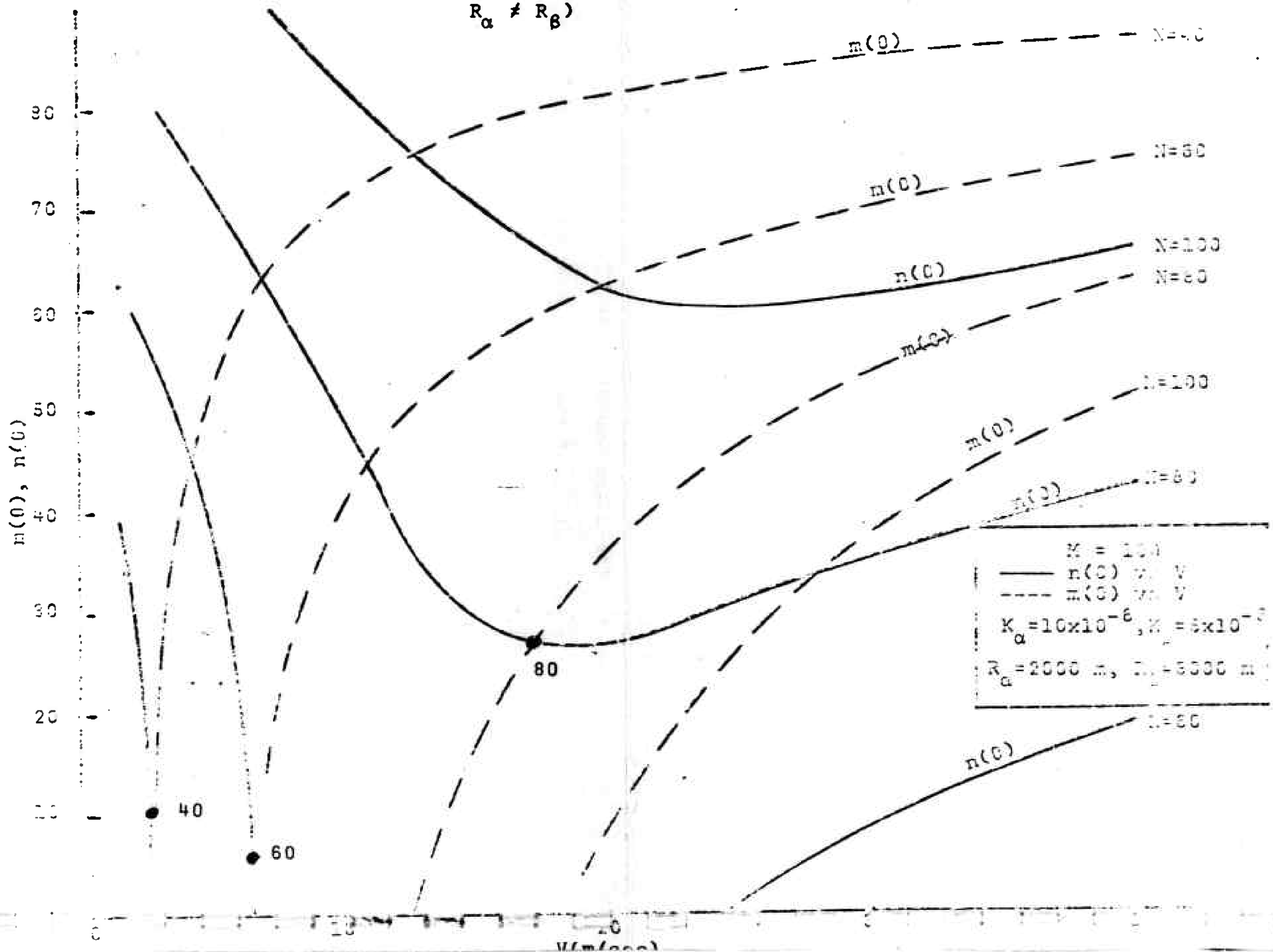


Figure 13 Force Difference at Range r = 0 (Linear Attrition Rate $R_\alpha \neq R_\beta$)

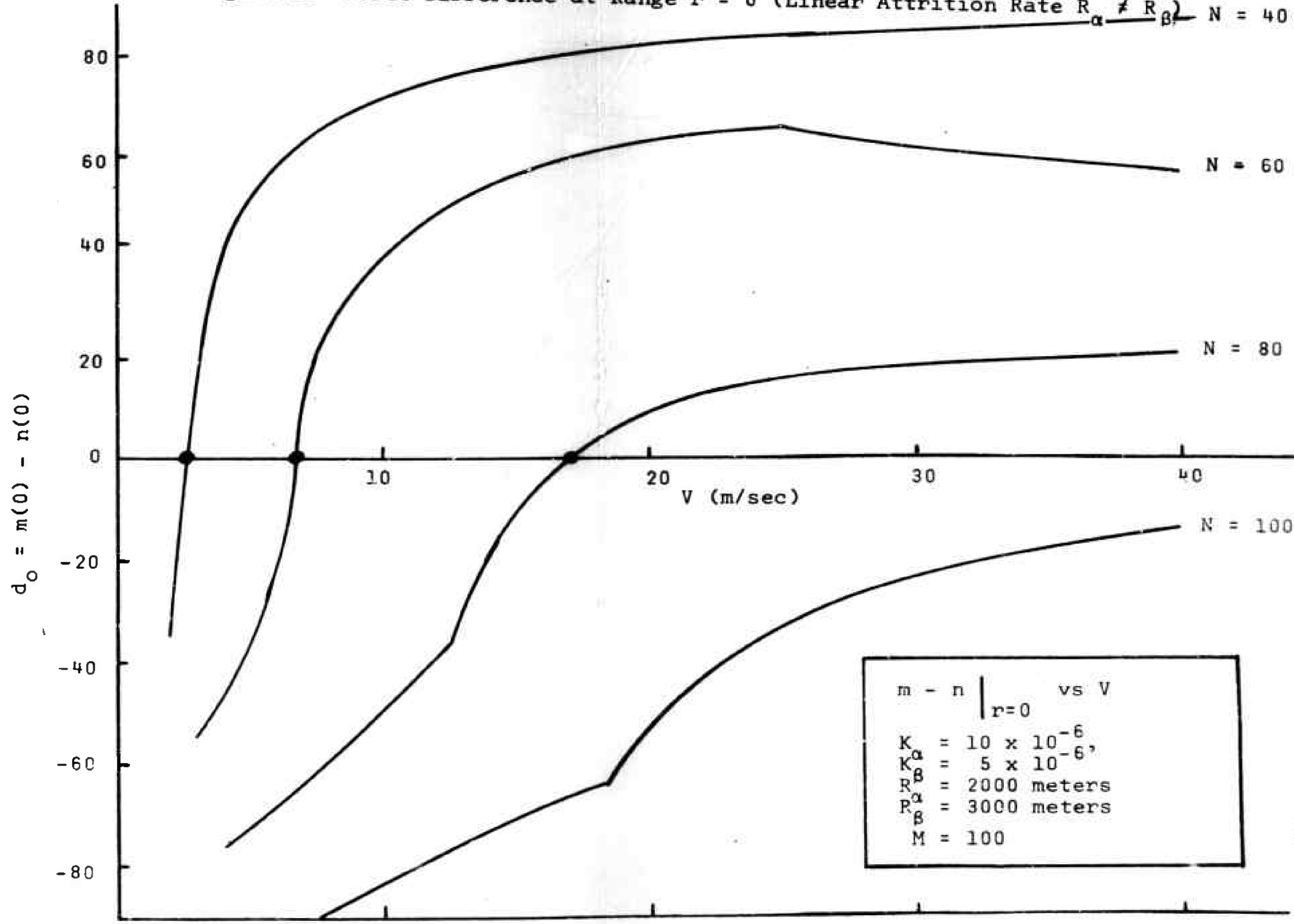
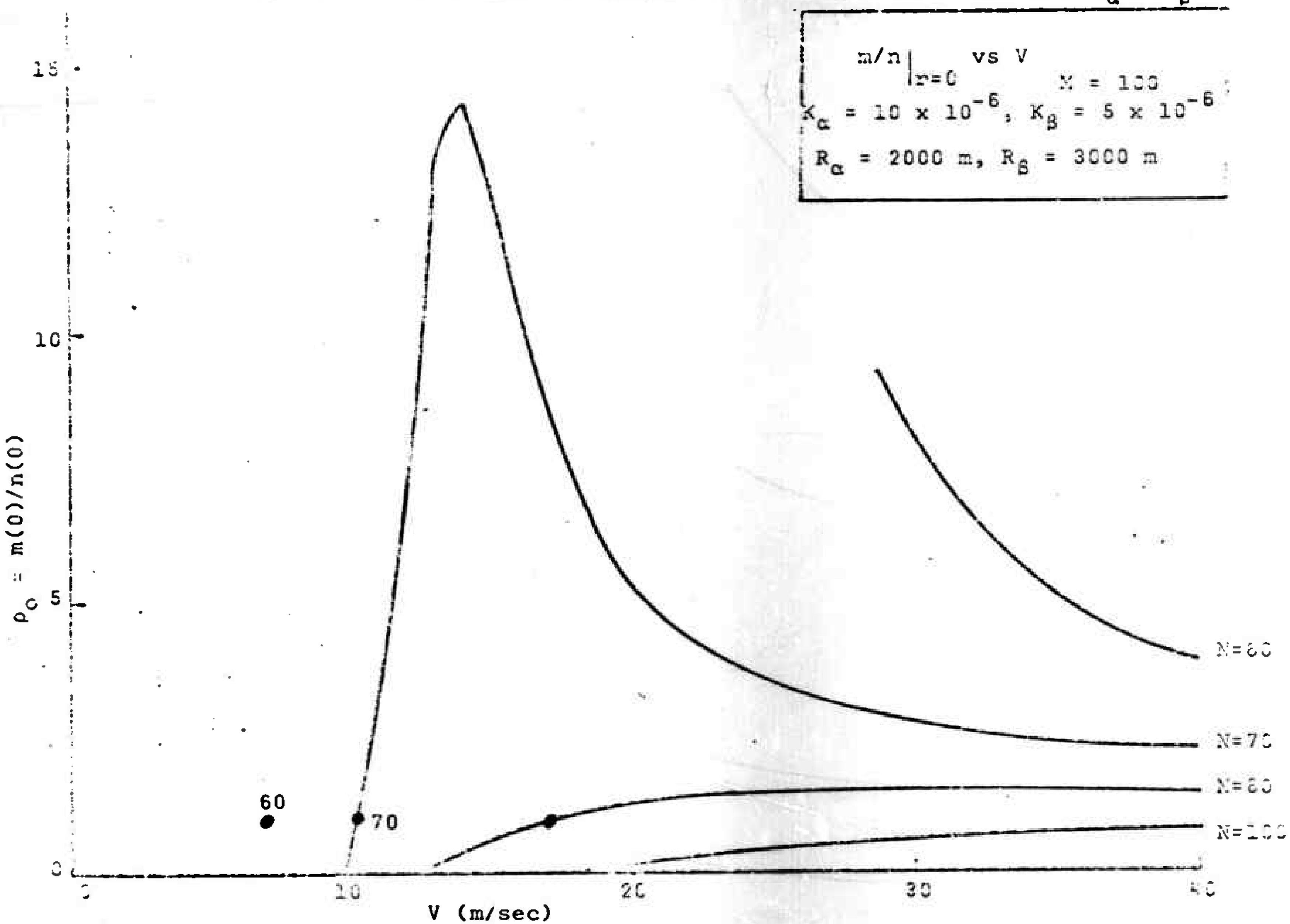


Figure 14 Force Ratio at Range r = 0 (Linear Attrition Rate $R_\alpha \neq R_\beta$)



This point is highlighted in Figure 14, where for an initial ratio of Blue to Red units of 1.43 (100 Blue to 70 Red), an assault speed of 15 meters/second produces a final ratio of forces of 14.5 to 1. A reduction of only 5 meters/second assault speed would produce a final ratio of Blue to Red forces of less than 1.0.

4.2 *Fire-Support Engagement Results*

Based on the previous results, it was recognized that the range dependency of the ratio of the attrition-rate functions was a major factor in inhibiting analytic solution procedures. Accordingly, to obtain some solution insights, a hypothetical "fire-support" situation was developed for which the Blue attrition-rate function was a constant and the Red weapon attrition-rate function was a linear function of range. This combination of attrition-rate functions leads to their ratio being range dependent but the equations are amenable to analytic solution. The tactical situation is shown in Figure 15 and depicts

1. a Red force (n) defending a fixed position at $r = 0$;
2. a Blue force (m), under fire from the Red force, moving from $r = R_O$ (range at which the battle begins) to $r = R_S$ at a constant speed (v) without returning fire on the Red force;

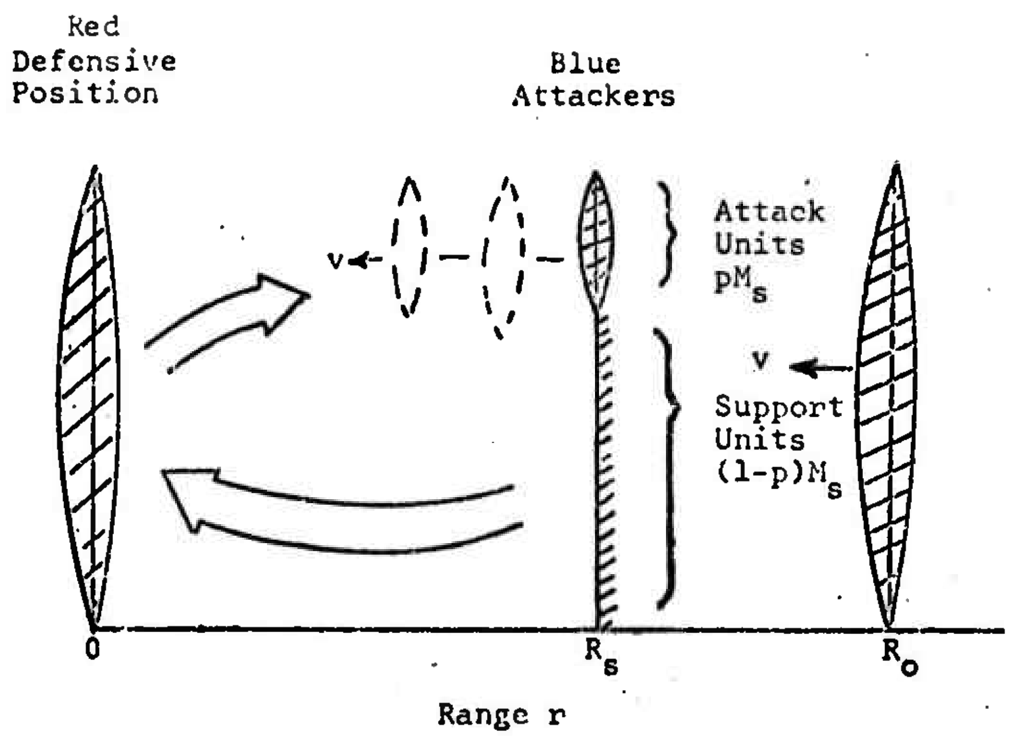


Figure 15 Fire-Support Tactical Situation

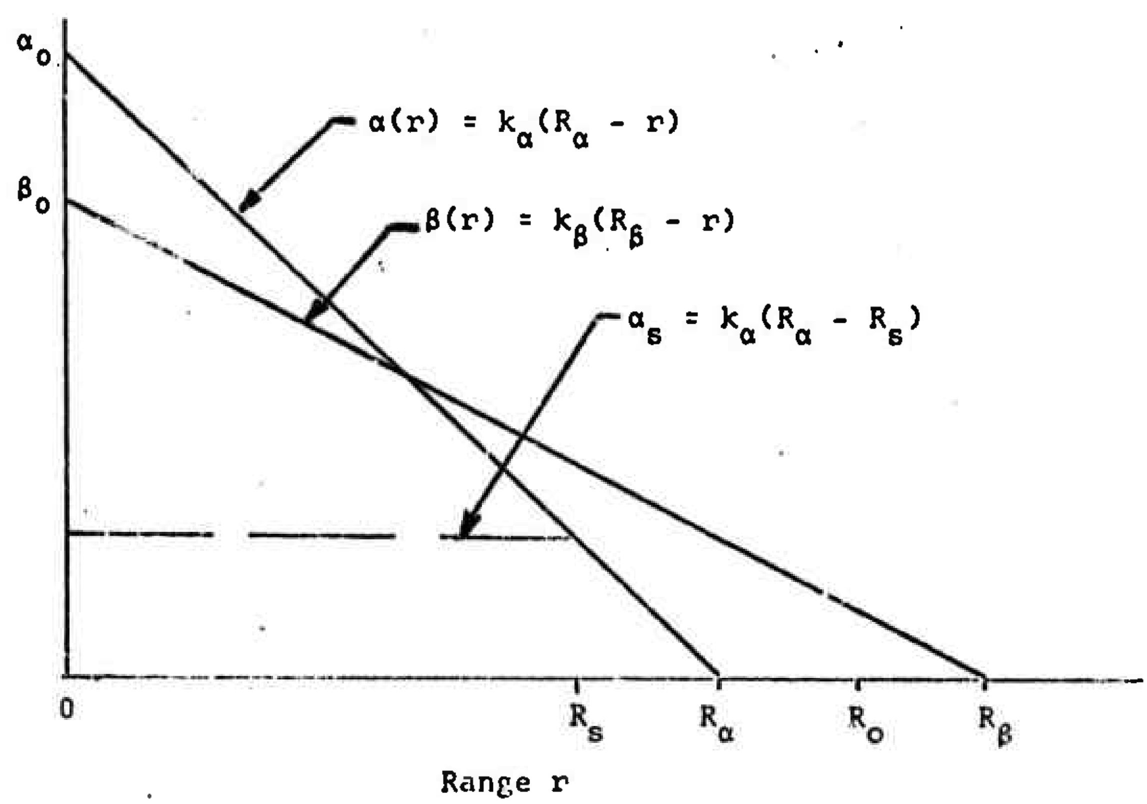


Figure 16 Attrition-Rate Functions for the Fire-Support Situation

3. at $r = R_s$, a percentage p of the remaining Blue force (M_s) continues to advance at speed v without firing. The remaining $(1 - p)M_s$ Blue units stop and provide supporting fire on the Red force.

4. Red fires only on the moving Blue units.

The attrition-rate functions which result from this situation are shown in Figure 16. The Red force attrition rate varies with range since Red units engage closing Blue units. The Blue attrition rate is a constant, $\alpha_s = k_\alpha(R_\alpha - R_s)$, since the supporting fire Blue units remain a fixed distance, R_s, from the Red units. Solutions to the resultant differential equations have been obtained and some analysis of optimal tactics (assault speed, percent force split, etc.) conducted. This work is described in [C, 6].

4.3 Heterogeneous-Force Results

A long-range objective of the research program is to obtain usable analytic solutions to the sets of variable-coefficient differential equations used to describe combat among heterogeneous forces. These are equations 1 and 2 in Chapter 2. The preceding sections discussed research to obtain solutions for simplified forms of these equations for homogeneous forces and a fire-support situation which retained the complexity of the variable attrition-rate functions. Research has been conducted on another form of simplification

in which we retain the generality of heterogeneous forces, but consider the attrition-rate functions to be independent of range for all weapons in the battle.

Previous research efforts in this area (Snow, 1948) developed solutions for this situation under the assumption that each Blue group distributes its fire over all Red groups and each Red group distributes its fire over all Blue groups. That is, the allocation factors $e_{ij} > 0$ and $h_{ji} > 0$ for all i and j. This assumption appears to be highly unrealistic in that it requires ineffective weapons to fire at targets they cannot destroy (a rifle firing on an armored tank) and an over-allocation of firepower (a long-range missile firing at an infantryman).

A general solution to the heterogeneous-force, constant attrition-coefficient battle model for any allocation policy has been developed. The solution methods are simplified, and thus more useful for analysis purposes, when the optimal zero-one allocation strategy is employed.

General analytic solutions to the heterogeneous-force, variable-coefficient battle models could not be developed. A numerical procedure was developed to solve the equations for simplified tactical situations in which the heterogeneous combat groups may have different locations and where the variation in attrition coefficients with range is explicitly

considered for each group. This procedure, which is described in [D, 3.0], was developed primarily for use as a research tool.

Chapter 5

RELATED RESEARCH RESULTS AND FUTURE NEEDS

Seth Bonder

The research described in this report is viewed as the beginnings of research activity to develop analytical models of relevant military processes that can efficiently and effectively be used in analysis of both small and large-scale military activities. This long-range objective will require the development of analytic structures for each of the relevant military processes (such as combat, reconnaissance, logistics, etc.) and research on methods of combining them into an integrated set of analytic procedures.

Modeling emphasis to date has been directed to the development of differential models of the combat process and associated allocation strategies. This chapter summarizes some related modeling results developed under the cited contracts and lists a few areas deemed important for future research.

5.1 Preliminary Modeling of Surveillance Patrols

Except for the intelligence factor included in the combat model structure, the differential models of the combat activity essentially ignore the intelligence-gathering or reconnaissance process that could reasonably have a large effect on combat effectiveness predictions, especially when

one considers its interaction with the allocation strategy. It was thought that many of the existing search and reconnaissance theories would be useful for predicting the amount of intelligence-gathering capability possessed by a tactical unit. A thorough literature search in this area, however, indicated that existing theories are less than useful for this purpose (Moore, 1970). Most of the research efforts have been devoted to a development of strategies for the optimal allocation of search effort and little to the development of descriptive models of intelligence-gathering processes nor its interaction with the combat activity, i.e., "subsequent action." The existing results do not consider important aspects such as intermittent target visibility, multiple targets, moving targets, and others. Accordingly, a small part of the research effort was devoted to the development of preliminary models of the intelligence-gathering process, specifically surveillance patrols.

The surveillance situation modeled is shown in Figure 17, where

v = speed of movement between subareas,

A = total area searched,

a_i = area of i^{th} subarea searched,

d_i = distance between subareas $(i - 1)$ and i,

n = number of subareas searched.

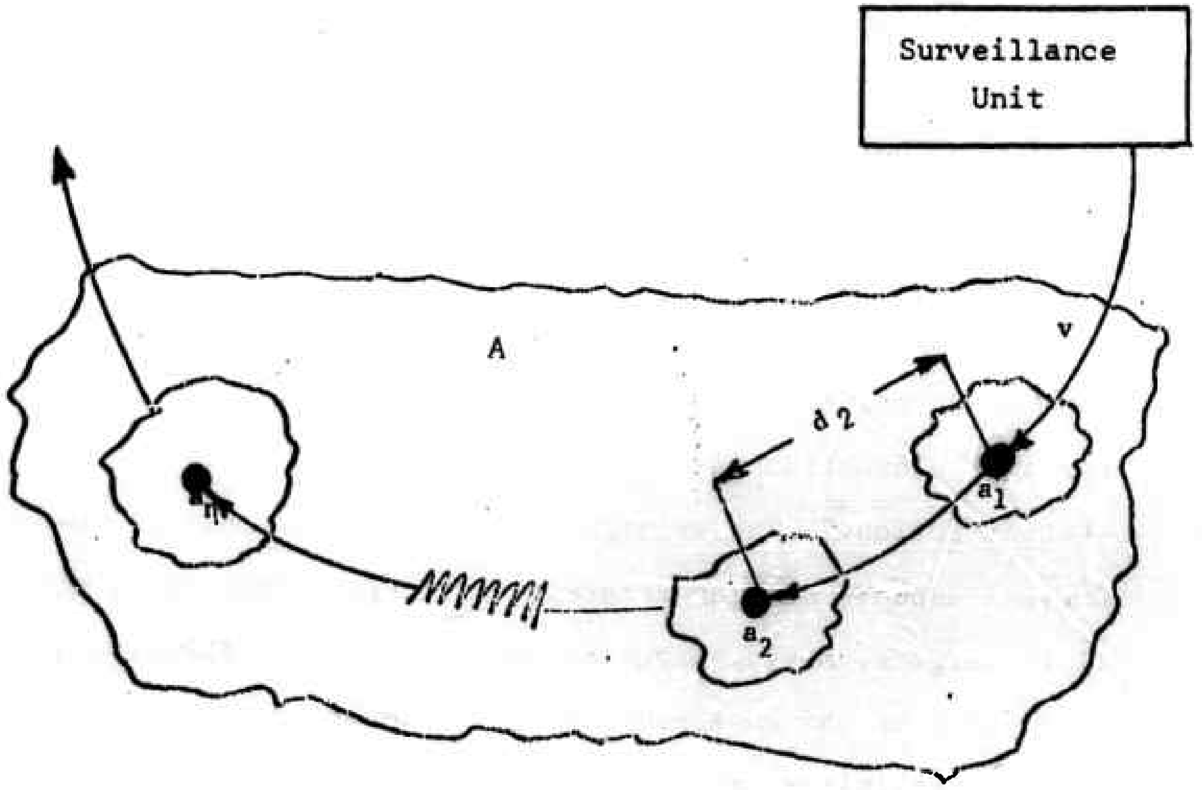


Figure 17 Surveillance Patrol

Search in successive areas A may be considered continuous search associated with a mobile force situation. Search in just one area A might be considered a periodic area surveillance to obtain general information during a static situation.

The models were developed on the assumption that the surveillance unit moves into a subarea and, as a unit, scans the area as a single sensor. The patrol leaves a subarea and goes

to another at the time it detects the target's presence or after a specified time during which it has not detected a target.

A number of models of the surveillance activity noted above were developed, each differing in assumptions regarding the stochastic nature of the visibility process (existence of line-of-sight). Mathematical expressions were developed for

(a) the probability of detecting a target in a subarea,

(b) the probability density function (pdf) for the time to detect a target in a subarea, given it is detected,

(c) the pdf of the time spent in a subarea,

(d) the pdf of the time until the first detection,

(e) the probability of detecting a target during the patrol,

(f) the pdf of the number of targets detected, and

(g) the pdf of the time spent searching the total area.

These expressions explicitly include the target's location, effects of the sensor capabilities, mobility of the sensors, and the line-of-sight disturbances of the terrain. The mathematical developments are described in [E, 2].

Modeling effort was also directed to the development of general mathematical structures to describe the visibility (line-of-sight) process. The model developed considers multiple periods of intervisibility between the sensor and the target and contains formulae for

(1) the probability that the target is visible for a given time t;

(2) the pdf of the length of time that a target will remain visible, given that it is visible at t;

(3) the pdf for the number of times the target will be visible in a fixed interval t_s;

(4) the pdf for the total time of visibility in t_s;

(5) the pdf for the number of visible targets at time t if there are N independent targets;

(6) the probability density function for the number of sightings in $(0,t_s)$ if there are N targets.

This work is described in [E, 3].

5.2 *Stochastic Duels with Reliability and Mobility*

The development of the differential models of combat extended the earlier Lanchester formulations to include mobility of both forces, microscopic details of the weapon systems in the attrition coefficients, and the fact that the attrition coefficients vary when forces employ mobile weapon systems. This approach was taken based on the judgment it would be more difficult at this time to enrich the stochastic duel theories (which already considered microscopic weapon system parameters) to include more than single duelists and simultaneously consider mobility of these forces. A small effort, however, was devoted to extending the "one-on-one" stochastic duel descriptions to include reliability

of the duelist's weapons and initial elements of mobility. This research is described in [F, 1].

Previous work in stochastic duel theory included some natural limitations of weapon systems in duels involving limited ammunition supplies and time limits. Another natural limitation of a weapon is the reliability of its firepower. The denigration of a weapon may be due to factors such as severe natural environment, lack of preventive maintenance, and the use of the weapon when fired. The first two factors concern the study of reliability and maintenance per se, while the third factor is more complex, since more than the temporary loss of firepower is at stake in combat.

Models were developed to describe catastrophic failures of firepower, leaving the duelist entirely helpless or forcing him to withdraw from the duel. Reliability is treated both as a function of time and as a function of the number of rounds fired, the latter as a more realistic model which relates the chance of breakdown to actual use of the system. The probability of one side winning is found for all the duels, and the results are compared with those for the corresponding "fundamental" stochastic duel.

A simplified model was developed to reflect the effect of mobility in a stochastic duel. The model incorporates single-shot kill probabilities that vary with time--the time

dependence occurring due to the basic dependence of accuracy and lethality on range to the target and the range variation due to movement of the weapon systems during the duel.

5.3 *Future Research*

During the course of research effort described in this report, it has become increasingly clear that research in other closely related areas will have to be performed in order to develop a reasonably complete spectrum of analytic models for defense planning. A brief description of some of these areas is given in this section.

Reconnaissance Research

A small amount of research effort was devoted to the development of preliminary mathematical structures of surveillance patrols which include effects of the visibility process, sensor detection capabilities, and mobility of the sensor system. It is felt that this work should continue to make the models more realistic of the reconnaissance process[1] and to determine optimal search strategies when explicit consideration is given to intermittent line-of-sight. Additionally, research should be directed to the problem of interfacing the reconnaissance activity with "subsequent action," primarily the combat activity. Questions that need

[1]See [E, 2.5] for suggested areas of enrichment.

be considered in this area include

(a) What model structures are needed to interface the reconnaissance activity and subsequent action?

(b) Can the effect of "false alarms" be effectively included in models of the reconnaissance activity when subsequent action is considered?

(c) What effect will consideration of subsequent action have on the optimal allocation of search effort?

Large-Scale Unit Modeling

Although the long-range objective of the research program is to develop models for both the microscopic weapon system planning problem and the macroscopic one of force structuring, initial efforts have been devoted to describing the microscopic structure of combat. Models to predict the attrition coefficients are being developed from elemental characteristics of individual weapon systems. These are then used as distinct parameters in the heterogeneous-force model for each i group in the Blue force and each j group in the Red force. There appear to be problems of size in using these models for large-scale force structuring[1] due primarily to the large number of dimensions in the formulation, i.e., consideration of only the attrition rates gives rise to I·J dimensions. Therefore, research is needed to determine the following:

[1]In contrast to the small unit composition of mixes of weapons systems.

1. The direct application of the heterogeneous differential equation formulation to large-scale force structures by reducing the dimensionality of the model. Methods would have to be developed to aggregate the attrition coefficient for different weapon groups to attrition coefficients for tactical units, which would then be used as input to a large-scale heterogeneous-force formulation.

2. Develop means of using the output of the microscopic heterogeneous model (which uses attrition coefficients for individual weapon groups) as input to other, perhaps differential equation type, models of large-scale force combat activities.

Close-Combat Research

The models currently under development will provide predictions of four basic dimensions of combat--time, space, casualties, and resources expended. Usually, some measure of effectiveness such as the ratio of survivors, the difference of survivors, and the percentage of survivors, at ranges close to the objective is computed and used as an indication of whether or not the combat activity was "successful."[1] However, little is known regarding the correlation between these measures and successful accomplishment of a mission.

[1]Examination of Figures 13 and 14 indicates that recommended assault speeds would differ for the difference and ratio measures of effectiveness.

Accordingly, it is felt that research is needed to assess the predictive capability of these measures for different combat activities.

Approximations to Variable-Coefficient Formulations

As shown in the solutions to the homogeneous-force models with variable coefficients, variation in the attrition coefficients during a battle appear to have a significant effect on the battle results. During some of the applications of the differential combat model in the Main Battle Tank program, however, it was found that in some situations the results of battle could well have been predicted with a constant-coefficient model of the battle activity. Accordingly, it is of research interest to see if an appropriate "average" attrition rate over all ranges of a battle can be determined which, when used in a constant-coefficient formulation, would produce similar results to the variable-coefficient heterogeneous-force models. The constant-coefficient heterogeneous-force analytic solutions developed to date can be used in this study.

Logistics Research

The models described in this report can be used to give an indication of the logistic support requirements for ammunition and POL. The time-to-kill probability distributions

are developed from the more fundamental distribution of the number of rounds required to defeat a target. Thus, there exists a means of determining the amount of ammunition required to obtain a specific level of combat effectiveness predicted by the differential combat models. Since the latter also include the spatial distribution of forces and their maneuver during engagements, POL requirements can be determined from the specific capabilities of vehicles employed. Thus, the models assume an infinite inventory of ammunition and POL with no constraint on the combat activity. Research should be directed to developing an explicit logistics model which can be integrated with the combat formulations to reflect logistics restraints on the combat activity.

Mobility Research

The effect of the mobility of combat units is considered in the differential equation formulations in a rather restrictive sense by examining the effect of mobility during the engagement. This might more appropriately be called the effect of maneuver, with mobility being reserved for the strategic aspects of transporting the units to the battle area. Clearly, in the structuring of large-scale forces, the planners must trade off the firepower and maneuver capabilitities of units and the ability to transport them to threat areas as

required. It is felt that analytic models of mobility that can be interfaced with models of combat between large-scale forces are needed.

Command and Control Research

As noted in the earlier discussions of the combat model, the allocation strategies being developed assume not only perfect intelligence but also perfect command and control. That is, given one determines optimal allocation strategies, can the command-control system implement the assignment policies? Research in this area should be directed to determining

1. how to reflect imperfect command and control in the combat model formulation, especially in its interaction with the allocation policies, and

2. how to predict the amount of command-control capability possessed by a tactical or strategic unit.

Intelligence Research

The differential models of combat include an intelligence factor as one of the elements in the attrition coefficient. This factor is included to account for the loss in efficiency (effectiveness) of a firing weapon when it is firing on either targets already attrited or on areas that are void of targets. A model was developed to predict the intelligence factor

(see equation 8, page 29); however, methods of estimating only one of its input parameters--the expected time to fire on a live target--are available. Research is needed on methods to estimate the other parameters of the intelligence factor model.

Appendix A

TEST OF THE GENERAL MODEL

Seth Bonder and Robert Farrell

As noted in the introductory chapter, the objective of this research program is the development of analytic models for defense systems planning. Chapters 2, 3, and 4 summarized the basic structure used to describe the combat process, the development of models to predict inputs to the structure, and research efforts to obtain analytic solutions to the combat formulations. Conceptually, one may view all the results described earlier as hypotheses or theories that need be verified against actual data, or at least compared to the results of detailed Monte Carlo simulations.

Under a separate contract with the Directorate, Weapon Systems Analysis, Office, Assistant Vice Chief of Staff, U.S. Army, a study was conducted to compare the combat predictions generated by the differential model of combat to those predicted by more detailed Monte Carlo simulation methods.[1] Under this study, the general heterogeneous-force model with variable attrition coefficients was applied to a set of

[1]This study was conducted by Vector Research, Incorporated, whose principals developed the methods described in this report.

tactical situations used in the TATAWS-III study, which is part of the overall Main Battle Tank (MBT-70) study program. The Individual Unit Action (IUA) Monte Carlo simulation of ground combat was used to evaluate candidate main battle tank systems and force structures of proposed battalion task forces.

Figure 18 depicts one of the tactical plans considered in the Main Battle Tank program to which the differential model of combat was applied. The tactical plan shown is a Blue attack engagement against a fixed Red defensive position. The attack is conducted along three major axes with four individual routes of advance per axis. Each route consists of individual main battle tank candidates and/or supporting armored personnel carriers equipped with rapid-fire weapon systems. In addition to these maneuver units of main battle tanks and personnel carriers, the Blue attack force had long-range missiles and short-range missiles, shown in the figure. The defending force is comprised of tanks, missiles, and armored personnel carriers equipped with rapid-fire weapons systems.

The Monte Carlo simulation of this engagement considered the movement, acquisition, and combat activity (duels) of each and every unit in the battle.[1] Maneuver, in terms of

[1] Some of the engagements considered as many as 100 individual weapon systems.

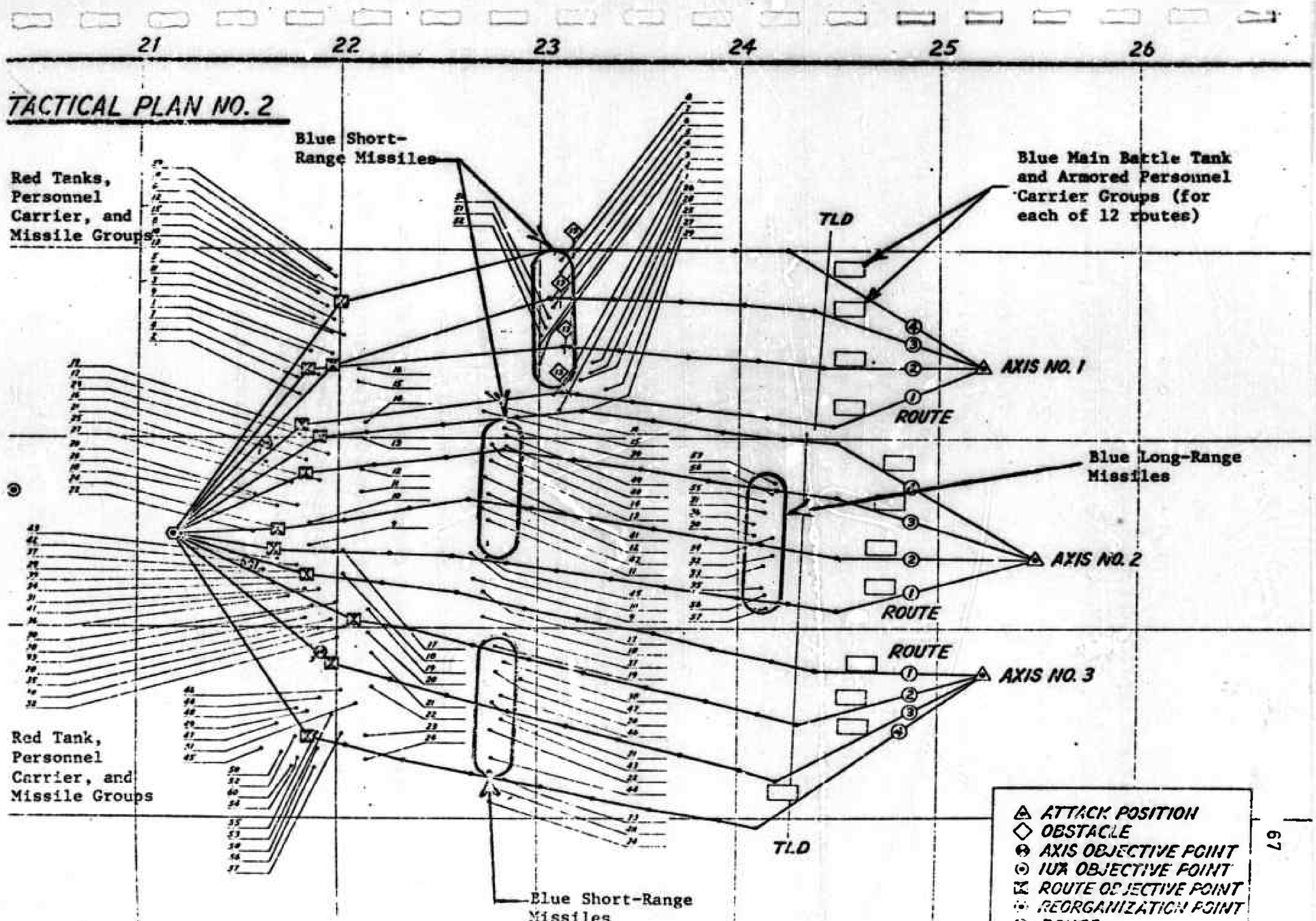


Figure 18 Tactical Plan for Blue Attack Engagement

attack speed and accelerations, over different portions of the terrain was considered for each weapon, based on pre-pocessed terrain analysis. The existence or nonexistence of line-of-sight between weapons systems for each route to all other weapons systems was used as input. Preprogrammed target priority tables were used to specify the allocation of individual weapons to targets. A replication of the simulation consisted of moving each of the systems down their prespecified paths and evaluating by Monte Carlo means the acquisition and attrition process (the fundamental duel event) for each weapon system during the course of the engagement. The engagement was replicated many times to obtain a level of statistical stability for the results.

The heterogeneous-force differential combat model was applied to this and other engagements by aggregating individual weapons systems into groups. Thus, for each route on an axis there were two separate groups of main battle tanks or armored personnel carriers. The long-range missiles were aggregated into one group and the short-range missiles were aggregated into three groups, one for each axis. The Red defensive force was aggregated by weapon type for each axis, thus producing nine Red defensive units. Also included, but not shown in the figure, were indirect-fire artillery weapons systems for both forces.

The attrition coefficients for each group on appropriate target groups were calculated using the same basic acquisition, firing time, accuracy, and lethality data used in the simulation. The coefficients were computed at 250-meter increments to the target out to a maximum range of 3,000 meters and stored as attrition-coefficient lookup tables. The allocation factors (e_{ij} and h_{ji}) employed were based on the priority tables used in the simulation.[1] The intelligence factor was set equal to 1.0 since these effects were not considered in the simulation.

Mobility and line-of-sight were considered in a deterministic manner similar to that employed in the simulation. Average speeds and lines-of-sight over segments of the routes were input for each of the aggregated groups. Thus, a group was moved as a whole, and visibility did or did not exist to the group as an entity.

It was noted in Chapter 4 of the text that closed-form solutions to the general heterogeneous-force, variable-coefficient, differential-equation model do not exist. Accordingly, the equations were solved numerically using the precomputed attrition coefficients and prespecified allocation factors which were stored as lookup tables.

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[1]A separate acquisition model was developed to estimate the percentage of surviving targets that were detected and, accordingly, could be allocated fire.

Using this approach, the model was applied to short-range defense and long-range attack engagements considered in the Main Battle Tank study program. Using these engagement types, six separate runs involving different weapon systems and force structures were made for comparison with the simulation results. These comparisons are shown in Tables 1-3.

Table 1 presents a comparison of the results of one of the short-range defense engagements. The initial numbers of forces and the numbers of survivors at three analysis points as predicted by both Monte Carlo simulation and the analytic model are given. The analysis points are defined by the percentage of Red tank survivors: low equal to 70 percent, principal equal to 50 percent, and high approximately equal to 20 percent. The times at which these analysis points are reached in each of the models is also given. Two sets of results at the low analysis point in the analytic model are shown since there was an appreciable attrition in the 240-250 time interval.

Table 2 presents the comparisons of tank survivors at the three analysis points for the other or short-range defense engagements, and Table 3 presents the comparisons of the tank survivors at the three analysis points for the

Table 1

COMPARISON OF SURVIVING FORCES

Run Number 7306

Short-Range Defense

Initial Numbers

16 Blue Tanks	40 Red Tanks
6 Blue Short-Range Missiles	0 Red Missiles
6 Blue APC	12 Red APC
3 Blue Long-Range Missiles	

ANALYSIS POINT	WEAPON	TATAWS SIMULATION	TIME	ANALYTIC	TIME
	Blue Tanks	13.90		15.1/13.9	
	Blue SR Missiles	5.10		6.0^{-}	
Low	Blue APC	5.93	242	6.0^{-}	240/250
	Blue LR Missiles	2.73		3.0^{-}	
(70%)	Red Tanks	28.00		30.4/24.4	
	Red Missiles	-----		--------	
	Red APC	11.70		11.0/10.6	
	Blue Tanks	12.23		12.6	
	Blue SR Missiles	4.57		6.0^{-}	
Prin-	Blue APC	5.73	263	6.0^{-}	260
cipal	Blue LR Missiles	2.27		3.0^{-}	
(50%)	Red Tanks	20.00		19.2	
	Red Missiles	-----		--------	
	Red APC	10.33		10.2	
	Blue Tanks	9.40		10.0	
	Blue SR Missiles	2.97		5.8	
High	Blue APC	5.20	327	6.0^{-}	290
(22%)	Blue LR Missiles	2.00		2.9	
	Red Tanks	8.90		7.2	
	Red Missiles	-----		--------	
	Red APC	4.27		7.0	

Table 2

COMPARISON OF SURVIVING FORCES

Run Numbers 7356, 7106

Short-Range Defenses

RUN NO.	WEAPON	INITIAL NO.	LAP		PAP		HAP	
			SIM.	ANAL.	SIM.	ANAL.	SIM.	ANAL.
7356	(Time)		(242)	(240)	(259)	(260)	(352?)	(280)
	Blue Tank	19	17.20	18.0	15.87	15.4	13.33	13.7
	Red Tank	40	28.00	30.4	20.00	17.6	8.0	8.8
7106	(Time)		(236)	(240)	(260)	(270)	(320)	(290)
	Blue Tank	27	20.13	18.27	15.1	11.6	9.37	6.7*
	Red Tank	40	28.00	26.00	20.00	19.2	11.07	15.2

*Red wins 60% of the time in this TATAWS game.

Table 3

COMPARISON OF SURVIVING FORCES

Run Numbers 7305, 7355, 7105

Long-Range Attack

			LAP		PAP		HAP	
RUN NO.	WEAPON	INITIAL NO.	SIM.	ANAL.	SIM.	ANAL.	SIM.	ANAL.
7305	(Time)		(206)	(260)	(411)	(440)	(512)	(470)
	Blue Tank	31	26.30	27.6	23.83	23.8	21.47	23.5
	Red Tank	13	9.00	9.1	7.0	7.0	2.0	1.4
7355	(Time)			(260)		(440)		(460)
	Blue Tank	37		32.6		29.2		29.2
	Red Tank	13		8.9		7.0		2.8
7105	(Time)		(415)	(340)	(477)	(480)	(610)	(540)
	Blue Tank	54	41.87	33.5	38.87	29.97	33.60	28.8
	Red Tank	13	9.0	9.0	7.0	7.0	2.0	2.4

three long-range attack engagements. The Monte Carlo simulation results for runs 7355 were not provided by the government for comparison. The larger differences in tank survivors in runs 7105 and 7106 were attributed to the fact that the input vulnerability data for the Blue tank on the Red missile used in the simulation run was approximately twice that used in the analytic model run.

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PART B

ATTRITION-RATE PREDICTION METHODS

The overall structure of the differential model of combat was presented in the preceding part of this report. A basic input to this model is the attrition rate, which is the rate at which a firing weapon system can destroy live targets when it is firing at them. This part of the report describes methods that have been developed to predict the attrition rate for a spectrum of weapon systems.

Chapter 1 describes our concept of the attrition rate. Rationale for employing the differential equation structure of combat (given in Part A) with this concept of the attrition rate, and an operational definition of the attrition rate for use in this context, is presented. Chapters 2, 3, and 4 contain descriptions of alternative developments of attrition-rate prediction models for various types of weapon systems. The attrition-rate models are developed using different mathematical approaches. Our intent is pedagogical, in that we hope it will acquaint the user with approaches to modify or develop attrition rates for systems other than those modeled in the research program.

Chapter 1

INTRODUCTION

Seth Bonder and Robert Farrell

1.1 Concept of the Attrition Rate

The attrition rate for individual weapon systems is assumed to be dependent on a multitude of physical parameters of a weapon system which describe its capabilities in such areas as acquisition, firing accuracy, delivery rate, and warhead lethality. Experience with existing systems suggests that these characteristics are dependent on the range to a target and are stochastic in nature. That is, the attrition rate is functionally dependent on the range between combatants and, for any specified range, is described by a probability distribution. In the vernacular of the mathematician, the attrition rate may be viewed as a nonstationary stochastic process when forces employ mobile weapons. This is shown in Figure 1, which depicts the two distinct variations in the attrition rate for a single weapon system type against one target type: (a) the stochastic variation at a specific range, which is described by the conditional probability distribution $f(\alpha|r)$, and (b) the variation in some function of the attrition-rate random variable with range, which is called the attrition-rate function, $\alpha(r)$.[1]

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[1]For clarity of discussion, variations in the attrition rate due to changes in target posture, environmental effect, etc., which can be included in the model, are not presented.

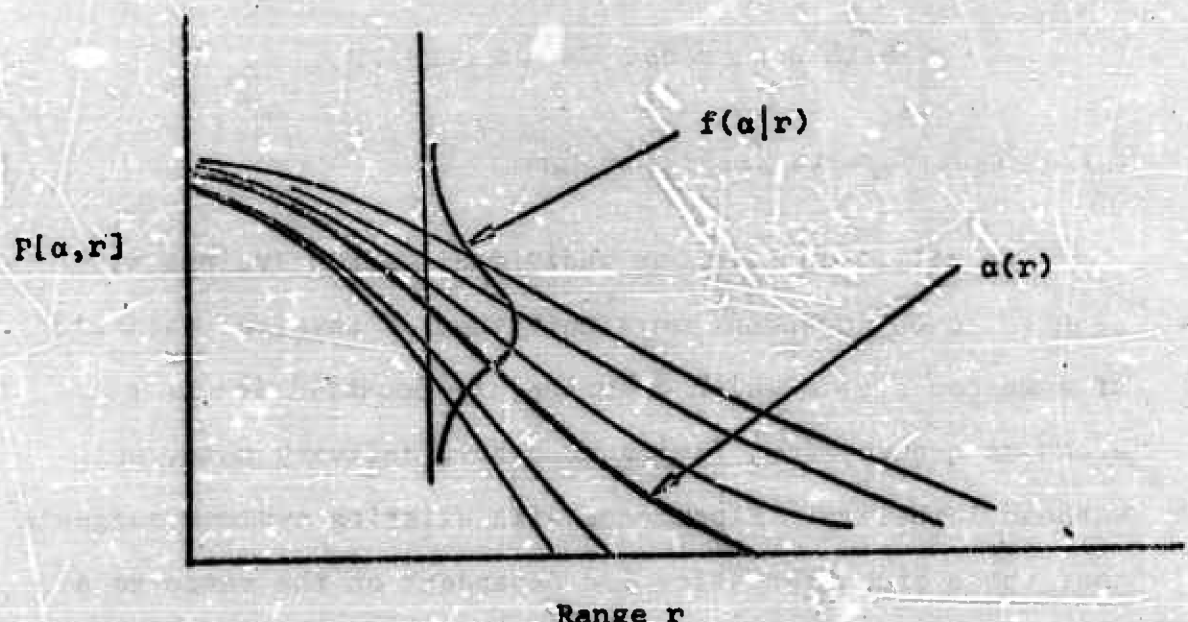

Figure 1 The Attrition-Rate Process

The fact that armed conflict is stochastic is well recognized and is one of the reasons for conceptualizing the attrition rate itself as a nonstationary stochastic process, $P[\alpha,r]$. Assuming the process $P[\alpha,r]$ could be predicted, one would like to incorporate the range and chance variations of the attrition rate explicitly into a model of combat among heterogeneous forces. The rate concept suggested that such a model would be either a differential equation (continuous-state variables) or a difference-differential equation (discrete-state variables) structure in which the relevant coefficients were nonstationary stochastic processes, i.e., the $P[\alpha_{ij},r]$ and $P[\beta_{ji},r]$ for all weapon-target group pairs. Initial study strongly indicated that,

in the foreseeable future, there was little hope of solving either of these structures even for simplified situations. A research decision was made to suppress the chance variation in the attrition rate and concentrate on structures of combat which explicitly involved the range variation in the rate when mobile weapons are employed.

Discrete-state stochastic process models were considered in which the transition rates are nonstationary, i.e., as varying with time. The literature indicated that discrete-state stochastic process formulations of combat have been difficult to solve even when the process is considered to be Poisson (Lanchester type) with stationary transition mechanisms. The few solutions obtained with homogeneous forces have been of such complexity as to delimit their usefulness for analysis purposes (Dolanský, 1964; Clark, 1968). Accordingly, it was felt that useful solutions for general discrete-state stochastic process formulations with nonstationary transition mechanisms could not be obtained in the near future.

Although the appropriate long-range objective is to develop stochastic formulations of heterogeneous-force armed combat such as those noted above, we felt that a more reasonable intermediate objective would be the development of deterministic formulations, and solutions, which included the nonstationary aspects of the attrition rate at the expense of

explicit consideration of its stochastic elements. Accordingly, the coupled sets of differential equations described in Part A of this report (equations 1 and 2), were chosen as the mathematical structure to model the combat activity. The nonstationary aspect of the attrition rates is included in the formulation as the variable coefficients in the differential equations, where the variable coefficients are appropriately defined as the attrition-rate function, $\alpha(r)$. Thus, there is one value of the attrition rate (for any firing weapon on a specific target group) at each range.

1.2 Definition of the Average Attrition Rate

Initially, the attrition rate at each range was defined to be the arithmetic mean or expected value of the attrition-rate random variable. Barfoot (1969) suggested that a more appropriate definition of the attrition rate, when a single value is used at a specific range, is the harmonic mean of the attrition-rate random variable. The appropriateness of this definition for use in the differential equation model of combat is seen below.

Consider a homogeneous-force battle in which the initial numbers of Blue (M) and Red (N) forces are sufficiently large so that neither is totally annihilated. Each Blue weapon system is engaged in a renewal process of attriting targets, i.e., the times between kills are independent and identically

distributed random variables. From Blackwell's theorem (Parzen, 1962, p. 183),

$$\lim_{t \to \infty} \Pr[\text{renewal in } (t, t + dt)] = \frac{dt}{\mu},$$

where

μ = the expected interrenewal time.

Therefore, the expected number of Red kills in (t, t + dt) is

$$E[\text{number of Red kills in } (t, t + dt)] = \frac{mdt}{\mu}. \quad (1)$$

The differential equation homogeneous-force model of combat states that

$$\begin{aligned} dn &= E[\text{number of Red kills in } (t,t + dt)] \\ &= \alpha m dt. \end{aligned} \quad (2)$$

Comparison of (1) and (2) suggests that α be defined as $1/\mu$. More generally, the definition of the attrition rate to use (for a specific range) in the differential equation structure of heterogeneous-force combat is

$$\alpha_{ij}(\text{at range } r) \overset{\text{def}}{=} \frac{1}{E[T_{ij}|r]}, \quad (3)$$

where

$E[T_{ij}|r]$ = the expected time for a single Blue system of the i^{th} group to destroy a passive j^{th} group Red target, given the target is at range r.

This definition for an average value of the attrition rate at range r is equivalent to the harmonic mean of the attrition rate when it is viewed as a random variable at range r. This definition also leads naturally to defining the range variation of the attrition rate as the variation in the reciprocal of $E[T_{ij}|r]$ as the range to the target changes. The range variation is called the *attrition-rate function* and is denoted by $\alpha_{ij}(r)$, as used in the differential equation structure of combat.

1.3 Taxonomy of Weapon Systems for Attrition-Rate Models

Because of the definition of the attrition rate given by (3), research on attrition rates has been concerned primarily with the development of *time-to-kill* probability distributions and their expected values for a spectrum of weapon systems. To ensure that the attrition rates developed are general, a taxonomy of weapons systems that is not dependent on physical hardware characteristics (such as caliber) was developed. Rather, the taxonomy reflects characteristics of weapons systems that would affect the methods used in predicting the attrition rates.

The taxonomy is shown in Figure 2. Weapon systems are first classified by their lethality characteristics as having either impact-to-kill mechanisms or area-lethality effects. Within each of these categories, we have found it useful to further classify weapon systems on the basis of their methods of using firing information to control the system aim point and their delivery

LETHALITY MECHANISM:

1. IMPACT
2. AREA

FIRE DOCTRINE:

1. REPEATED SINGLE SHOT:
   - *A) WITHOUT FEEDBACK CONTROL OF AIM POINT
   - *B) WITH FEEDBACK ON IMMEDIATELY PRECEDING ROUND (MARKOV FIRE)
   - C) WITH COMPLEX FEEDBACK
2. BURST FIRE:
   - *A) WITHOUT AIM CHANGE OR DRIFT IN OR BETWEEN BURSTS
   - *B) WITH AIM DRIFT IN BURSTS, AIM REFIXED TO ORIGINAL AIM POINT FOR EACH BURST
   - C) WITH AIM DRIFT, RE-AIM BETWEEN BURSTS
3. MULTIPLE TUBE FIRING: FEEDBACK SITUATIONS (1A, B, C)
   - *A) SALVO OR VOLLEY
4. MIXED MODE FIRING:
   - A) ADJUSTMENT FOLLOWED BY MULTIPLE TUBE FIRE
   - *B) ADJUSTMENT FOLLOWED BY BURST FIRE

* INDICATES THAT ANALYSIS OF THIS CATEGORY HAS BEEN PERFORMED.

Figure 2 Weapon System Classification for the Development of Attrition Rates

characteristics, i.e., the firing doctrine employed.

The first cases analyzed involved single-tube firings in which launch of a projectile occurred only after the observation of the effects of the preceding round. These are called "repeated single-shot" doctrines in our schema, and are sometimes called "shoot-look-shoot" doctrines by other analysts. Analyses have been undertaken of two subclasses: (a) those in which no use is made of information obtained from observations and (b) those in which the observations are treated distinctly depending on whether they are a hit or a miss, leading to different types of correction in aim point for these two cases. This subclass is called "Markov fire." Other more complex feedback situations have not been analyzed.

The more complex doctrines involving "multiple-tube firings" and "burst fire," have been analyzed separately. These are classes of systems for which the projectiles may be launched before observation of previous round effects. Burst-fire cases analyzed include those in which rounds are all identical with respect to accuracy (no drifting or controlled alteration of the aim point) and those in which the rounds within a burst vary, but the bursts are resighted to the same aim point. All present analyses have been based on fixed-length bursts. The complex case in which bursts are re-aimed on the basis of observation has not been analyzed. Preliminary analyses have been conducted of multiple-tube firing cases, and it has been determined that the attrition rate for both volley and salvo fire may be represented by the same formulae. The mixed-mode firing doctrine in which a period of

of single-shot fire is followed by burst fire has also been analyzed.

It is important to note that this classification scheme of weapon systems is not complete and that even in the areas where analysis has been conducted, the formulae developed do not necessarily represent all weapons systems in the appropriate category. Use of the attrition-rate formulae presented should be preceded by a careful comparison of the assumptions used in developing them with the lethality characteristics and firing doctrine of the weapon system being considered.

The succeeding chapters of this part of the report describe the detailed development of attrition-rate models for the different classes of weapon systems. The developments are organized by the mathematical assumptions and techniques used, and include multiple approaches in obtaining the same and similar results in some of the cases. Our intent is pedagogical, in that we hope it will acquaint the user with approaches to modify or develop attrition rates for systems other than those modeled in the research program.

Chapter 2 utilizes detailed probability analyses to determine the complete probability distribution of the time-to-kill random variable under the following assumptions:

(a) The systems are impact-lethality, repeated single-shot systems of class 1A or 1B,

(b) The probability of kill given an impact is identical for every round fired,

(c) The time preceding the firing of the first round is not random, and the conditional times to fire a round after a hit and after a miss are not random,

(d) The probability that a round fired after a preceding hit or miss results in a hit or a miss is not influenced by the knowledge of other history of the engagement (such as the number of rounds fired or the number of previous hits),

(e) The engagement terminates immediately on a kill.

This chapter also presents straightforward probability analyses of the expected time-to-kill in the impact-lethality burst-fire problem which do not involve calculations of the complete probability distributions.

Chapter 3 presents an alternative mathematical methodology for the development of probability distributions and expected values of the time-to-kill variables in the repeated single-shot impact-fire case. The method permits relaxation of assumptions (b) and (c) above, but involves the extensive use of Laplace transform analyses of random variables. Thus it is somewhat more general, but also more mathematically difficult, than the methods of Chapter 2.

Chapter 4 presents a very general method of determining the expected time to kill a target for a broad class of weapon systems which includes the repeated single-shot impact-lethality category. The methods used do not determine the full distribution of the time-to-kill random variable. The methods, although based in the theory of Markov-renewal or semi-Markov processes, do not require detailed understanding of the theory in its application. Only very general assumptions concerning the firing and lethality processes are required.

Chapter 5 describes the development of attrition-rate models for area-lethality systems. The methods are straightforward detailed analyses of the process, similar in general philosophy to the burst-fire analyses of Chapter 2, but differing in techniques. The analyses are based on previously documented models of the artillery fire process. This chapter does not specifically consider the kill rate in terms of the time-to-kill random variable.

1.4 *References*

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Chapter 2

IMPACT-LETHALITY SYSTEMS
REPEATED SINGLE-SHOT, BURST, AND MIXED-MODE FIRE DOCTRINE

Seth Bonder

This chapter presents the development of models to predict the attrition rate for many of the weapons classified as impact-lethality systems. Systems of this type aim at a point target and projectiles must impact upon the target to destroy it. Methods are developed for repeated single-shot, burst, and mixed-mode single-shot Markov and burst-fire doctrines. The results are models for the probability density function and the expected value of the *time-to-kill* random variable at a specific range to the target since, by definition, they are used directly to predict the attrition rate at a specific range. Although the conditioning on range is explicit in the basic definition of the attrition rate (see equation 3, Chapter 1), the range notation is omitted in the remainder of this part of the report for clarity of development. For similar reasons, the i,j notation for weapon-target pairs is also omitted.

2.1 Repeated Single-shot, Markov Fire Doctrine[1]

Consider first the development of an attrition-rate model for repeated single-shot, Markov fire weapon systems. Exposition of the development proceeds as a straightforward analysis of the physical process. Implicit in this type of development are several assumptions which are listed here as a convenient summary and reference. These are

(a) the systems are of the impact-lethality, repeated single-shot, Markov-fire class,

(b) the probability of kill given an impact is identical for every round fired,

(c) the time preceding the firing of the first round is not random, and the conditional times to fire a round after a hit and after a miss are not random,

(d) the probability that a round fired after a preceding hit or miss results in a hit or miss is not influenced by the knowledge of other history of the engagement (such as the number of rounds fired or the number of previous hits),

(e) the engagement terminates immediately on a kill.

A reasonable physical manifestation of the single-shot, Markov fire doctrine is given by a main tank gun whose firing

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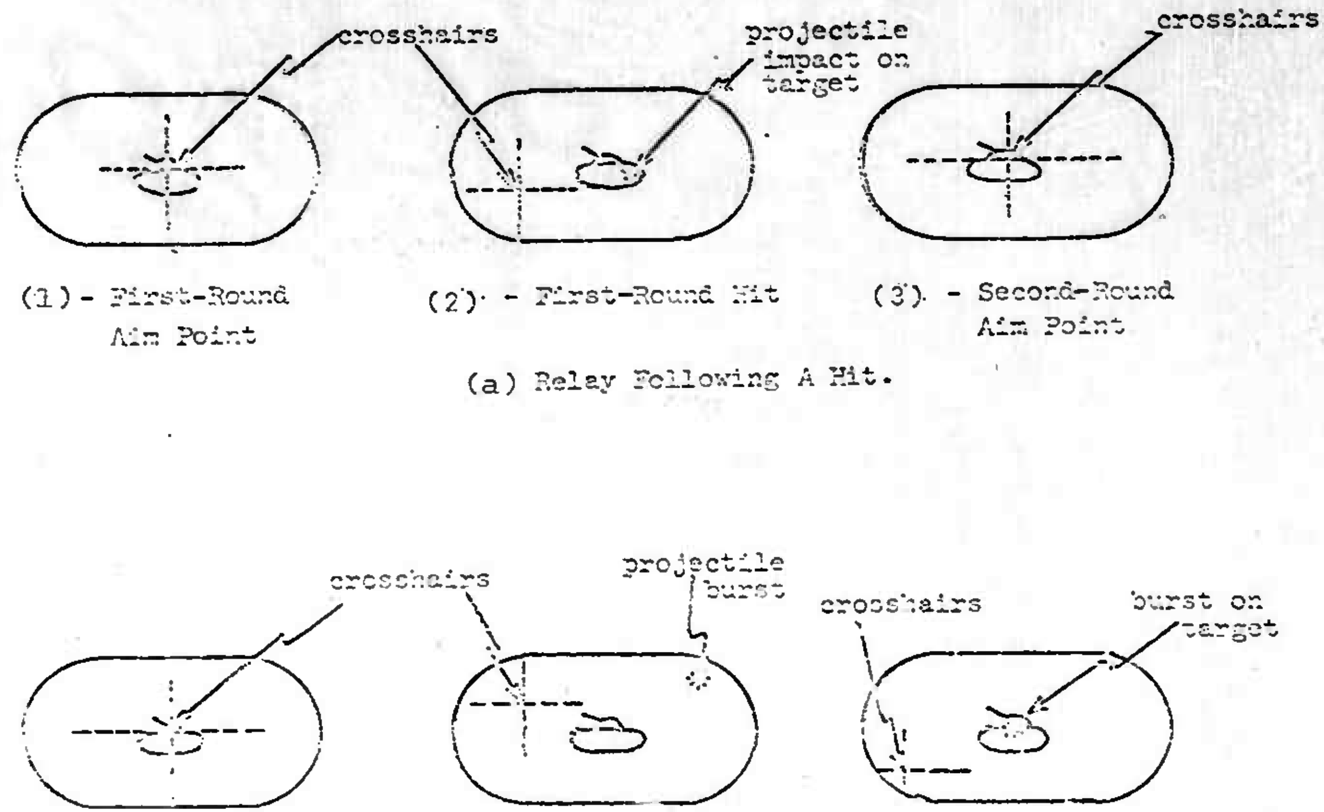
[1] Part of the derivation in this section is given by Bonder (1967), but are repeated here for convenience and continuity of development.

process is said to vary from round to round as shown in Figure 1. Figure 1(a) shows the adjustment procedure following a hit on the first round which is to replace the crosshairs on the target--presumably the position of the crosshairs for the first round. Figure 1(b) depicts the "burst-on-target" adjustment doctrine following a miss on the first round. Succeeding adjustments, based on the result of the immediately preceding round, are made in a similar fashion until the target is defeated. The probability density function (pdf) of the time to accomplish this result is obtained by essentially modeling this adjustment process as it occurs, round by round.

Since the objective of a weapon system is to defeat the enemy, we begin by defining lethality and its unit of measurement.[1] In brief, lethality refers to what happens to the target when struck by a projectile. The particular effect of interest is the target's combat utility. When this combat utility is reduced to zero, the target no longer poses an active tactical threat and may be considered defeated or killed. The definition of a defeated or killed target is, of course, dependent on the target's mission or role in combat. For example, consider an armored tank which is frequently referred to as "mobile,

[1]The lethality definition is paraphrased from Zeller (1961).



(1) - First-Round Aim Point

(2) - First-Round Hit

(3) - Second-Round Aim Point

(a) Relay Following A Hit.

(1) - First-Round Aim Point

(2) - First-Round Miss

(3) - Second-Round Aim Point (Burst on Target)

(b) Adjustment Following A Miss.

Figure 1 Firing Doctrine for a Main Tank Gun

protected firepower." Some of the tank's combat missions require primarily firepower, others require mobility, and still others require both firepower and mobility, and the definition of lethality must consider which of these are relevant in the context of a study.

Lethality against a particular target is measured as the conditional probability of a kill, given the projectile hits the point target, and noted symbolically as either $P(K|H)$ or P_K. This measure is dependent on the mechanical damage caused by perforating and/or striking the target, and the loss in combat utility resulting from this mechanical damage. Procedures developed to predict this measure for different types of targets have been developed. See, for example, Zeller (1961), Goulet (1963), Freedman (1965), and Meyer (1967).

Another measure of lethality can be defined as "the number of hits, z, needed to defeat the target." Since we are concerned with destroying the target just once, this measure is directly related to the conditional kill probability by the geometric density function

$$p(z) = (1 - P_K)^{z-1} P_K \ . \qquad (1)$$

The number of hits needed to defeat the target, z, is initially used as a parameter in subsequent developments of this chapter.

The number of hits required to effect a kill describes a weapon's lethality characteristics against particular targets. The weapon's accuracy capabilities are next considered by developing the distribution for the number of rounds fired (hits and misses) to defeat the target.

Let

P_1 = first round hit probability,

p = conditional probability of a hit given the preceding round fired missed the target,

u = conditional probability of a hit given the preceding round fired hit the target,

and consider the sequence of trials (rounds fired) connected in a regular Markov chain with transition probability matrix

$$\begin{array}{rc} & \begin{array}{cc} \text{hit} & \text{miss} \end{array} \\ \begin{array}{r} P_1 \text{ hit} \\ (1 - P_1) \text{ miss} \end{array} & \begin{pmatrix} u & 1 - u \\ p & 1 - p \end{pmatrix} \end{array} \qquad \begin{array}{l} 0 < u < 1 \\ 0 < p < 1 \,. \end{array}$$

It is assumed that p and u are defined only on the open interval (0,1). We seek the pdf for the number of rounds, N, to obtain z hits if the sequence of firings ends with a hit.[1] This can occur in two mutually exclusive and

[1]The procedure could be extended to remove this assumption that the firer recognizes when the target is defeated without technical difficulty but with increased complexity of discussion.

collectively exhaustive ways.

$$f(N|z) = f(N\cdot H\cdot H|z) + f(N\cdot M\cdot H|z) \; . \qquad (2)$$

The first term on the right-hand side of (2) is the probability that the first and last rounds of the sequence result in hits given that the z hits occur in N firings. The second term is the probability that the first and last rounds of the sequence result in a miss and a hit, respectively, given that the z hits occur in N firings.

To determine $f(N\cdot H\cdot H|z)$ we consider the following combination of firing results:

In the first r_1 firings, the event hit occurs everytime;
In the next s_1 firings, the event miss occurs everytime;
In the next r_2 firings, the event hit occurs everytime;
In the next s_2 firings, the event miss occurs everytime;
....
In the next s_{k-1} firings, the event miss occurs everytime;
In the last r_k firings, the event hit occurs everytime,

The joint occurrence of these events has the probability

$$P_1u^{r_1-1}(1 - u)(1 - p)^{s_1-1}pu^{r_2-1}(1 - u)(1 - p)^{s_2-1}p\ldots pu^{r_k-1}$$

$$= P_1u^{r_1+r_2+\ldots r_k-k}(1 - u)^{k-1}(1 - p)^{s_1+s_2+\ldots s_{k-1}-(k-1)}p^{k-1}. \tag{3}$$

Since there are a total of z hits and (N - z) misses,

$$\sum_{i=1}^{k} r_i = z \qquad \text{and} \qquad \sum_{i=1}^{k-1} s_i = N - z \ .$$

Therefore, (3) becomes

$$P_1u^{z-k}(1 - u)^{k-1}(1 - p)^{N-z-k+1}p^{k-1}$$

Accordingly, the probability of the outcome depends only on N, z, and k and not on the values of r_i and s_i. The number of hits, z, can be expressed as a sum of k positive integers (the r_i) in $\binom{z-1}{k-1}$ ways and the number of misses, (N - z), as a sum of (k - 1) positive integers (the s_i) in $\binom{N-z-1}{k-2}$ ways.[1] Therefore, the probability that it takes N firings to obtain z hits, the first and last being hits with probability P_1 and p or u, respectively--where the hits occur in k groups and the misses in (k - 1) groups--is

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[1]Proof of this assertion is given in Appendix B, 2, 1.

$$P_1 \binom{z-1}{k-1} \binom{N-z-1}{k-2} u^{z-k} (1-u)^{k-1} p^{k-1} (1-p)^{N-z-k+1} \ .$$

The outcome can occur for all values of k such that ($1 \leq k \leq z$). Accordingly,

$$f(N \cdot H \cdot H | z) = \begin{cases} P_1 u^{z-1} & N = z \\ P_1 \displaystyle\sum_{k=2}^{z} \binom{z-1}{k-1} u^{z-k} (1-u)^{k-1} p^{k-1} \binom{N-z-1}{k-2} (1-p)^{N-z-k+1} & \\ . & N > z \end{cases} \quad (5)$$

since $\binom{N-z-1}{k-2} = 0$ when $k = 1$ and $N > z$.

By an analogous derivation, it can be shown that

$$f(N \cdot M \cdot H | z) = (1 - P_1) \sum_{k=1}^{z} \binom{z-1}{k-1} u^{z-k} (1-u)^{k-1} p^k \binom{N-z-1}{k-1} (1-p)^{N-z-k} \quad \text{for } N > z \ . \quad (6)$$

Substituting (5) and (6) into (2) completes the derivation for

$$f(N|z) = \begin{cases} P_1 u^{z-1} & N = z \\ \left[P_1 \displaystyle\sum_{k=2}^{z} \binom{z-1}{k-1} u^{z-k} (1-u)^{k-1} p^{k-1} \binom{N-z-1}{k-2} q^{N-z-k+1} \right. & \\ \left. + Q_1 \displaystyle\sum_{k=1}^{z} \binom{z-1}{k-1} u^{z-k} (1-u)^{k-1} p^k \binom{N-z-1}{k-1} q^{N-z-k} \right] & N > z \ , \end{cases} \quad (7)$$

where $Q_1 = (1 - P_1)$ and $q = (1 - p)$. The reader is reminded that equation 7 is a conditional distribution which is dependent on the integer z.

The characteristic function of (7) is defined as

$$\phi_{N|z}(s) = E[e^{isN}] = \sum_{N=0}^{\infty} e^{isN} f(N|z) , \qquad (8)$$

where s is a dummy variable and $i = \sqrt{-1}$.

It is shown in Appendix B, 2, 2 that

$$\phi_{N|z}(s) = e^{isz} \left[P_1 + \frac{Q_1 pe^{is}}{1 - qe^{is}} \right] \left[u + \frac{(1 - u)pe^{is}}{1 - qe^{is}} \right]^{z-1} . \qquad (9)$$

Setting s = 0 in (9),

$$\phi_{N|z}(0) = \sum_{N=0}^{\infty} f(N|z) = 1 ,$$

proves that (7) is, in fact, a probability density function. The expected value of N is obtained from (9) as

$$E[N|z] = \frac{1}{i} \left. \frac{d\phi_{N|z}(s)}{ds} \right|_{s=0}$$

$$= z + \frac{(1 - P_1)}{p} + \frac{(1 - u)(z - 1)}{p} . \qquad (10)$$

The density function $f(N|z)$ for the number of rounds that must be fired to destroy a particular target is dependent on the lethality and accuracy capabilities of the weapon system. Two other important weapon characteristics remain to be considered--the system's acquisition capabilities and its rate of fire. We consider these characteristics in a manner such that the acquisition and firing processes are serial. That is, targets are destroyed by sequentially acquiring a target, attriting it by fire, acquiring a new target, attriting it, acquiring a new target, etc. This is in contrast to parallel acquisition and firing processes in which new targets may be acquired while a previously acquired one is being attrited.

We include the timing characteristics of acquisition and firing by defining

τ_a = the time to acquire targets,

τ_1 = time to fire the first round,

τ_h = time to fire a round given the preceding round was a hit,

τ_m = time to fire a round given the preceding round was a miss,

τ_f = projectile flight time,

and consider the following sequence of events from target acquisition to destruction. The sequence begins with detection which takes τ_a time units to occur. The first round is then fired and arrives at the target area $(\tau_1 + \tau_f)$ time units later. If the

first round misses, the next round will arrive $(\tau_m + \tau_f)$ time units after the first. If the first round hits the target, and more than one hit is required $(z > 1)$, the next round will arrive $(\tau_h + \tau_f)$ time units later. The sequence of firing after hits and misses is continued until the final hit which destroys the target is obtained. This description is consistent with our single-shot Markov firing doctrine in which the result of the previous round is observed before the next one is fired. In this process, rounds will be fired after each of $(z - 1)$ hits and $(N - z)$ misses. Accordingly, the time to defeat a target may be written as

$$T = \tau_a + (\tau_1 + \tau_f) + (\tau_h + \tau_f)(z - 1) + (\tau_m + \tau_f)(N - z)$$

$$= c_1 + c_2 N \;, \tag{11}$$

where

$$c_1 = \tau_a + \tau_1 - \tau_h + (\tau_h - \tau_m)z \tag{12}$$

$$c_2 = \tau_m + \tau_f \;. \tag{13}$$

Equation 11 defines T as a linear function of the discrete random variable N, and establishes a one-to-one transformation between their respective sample spaces. The density function of T is readily obtained from (7) by the change of variables technique for discrete variables (Hogg and Craig, 1959) as

$$f(T|z) = \begin{cases} P_1 u^{z-1} & T = c_1 + c_2 z \\ \left[P_1 \sum_{k=2}^{z} \binom{z-1}{k-1} u^{z-k} (1-u)^{k-1} p^{k-1} \binom{\left[\frac{T-c_1}{c_2}\right]-z-1}{k-2} q^{\left(\frac{T-c_1}{c_2}\right)-z-k+1} \right. & \\ \left. + Q_1 \sum_{k=1}^{z} \binom{z-1}{k-1} u^{z-k} (1-u)^{k-1} p^{k} \binom{\left[\frac{T-c_1}{c_2}\right]-z-1}{k-1} q^{\left(\frac{T-c_1}{c_2}\right)-z-k} \right]. & T > c_1 + c_2 z \end{cases} \tag{14}$$

The characteristic function of T, $\phi_{T|z}(s)$, is obtained directly from (9) by employing the following property of characteristic functions:

$$\begin{aligned} \phi_{T|z}(s) &= E\left[e^{isT}\right] \\ &= E\left[e^{is(c_1+c_2N)}\right] \\ &= e^{c_1 is} E\left[e^{isc_2N}\right] \\ &= e^{c_1 is} \varphi_{N|z}(c_2 s) \\ &= e^{is(c_1+c_2z)} \left[P_1 + \frac{Q_1 p e^{ic_2s}}{(1 - qe^{ic_2s})}\right] \left[u + \frac{(1-u)pe^{ic_2s}}{1 - qe^{ic_2s}}\right]^{z-1} . \end{aligned} \tag{15}$$

The expected value of T can be obtained from (15), or, more directly, by employing the linear property of the expected-value operator with (11). Accordingly,

$$E[T|z] = c_1 + c_2 E[N|z]$$

$$= c_1 + c_2\left[\frac{(1 - P_1)}{p} + \frac{(z - 1)(1 - u)}{p} + z\right]. \quad (16)$$

The characteristic function, $\phi_{T|z}(s)$, and the expected time to destroy a target, $E[T|z]$, are conditioned on the integer-valued lethality variable z, which is the number of hits required to destroy the target. This conditioning is removed and the continuous lethality parameter P_K (the conditional probability of destroying the target given it is hit by a projectile) introduced by the operations

$$\phi_T(s) = \sum_{z=1}^{\infty} \phi_{T|z}(s)p(z)$$

$$= \frac{P_K e^{is(\tau_a+\tau_1+\tau_f)}\left[P_1 + \dfrac{Q_1 p e^{isc_2}}{1 - qe^{isc_2}}\right]}{1 - (1 - P_K)e^{is(\tau_h+\tau_f)}\left[u + \dfrac{(1 - u)pe^{isc_2}}{1 - qe^{isc_2}}\right]},$$

(17)

where p(z) is given by equation 1 and

$$E[T] = \sum_{z=1}^{\infty} E[T|z]p(z)$$

$$= \tau_a + \tau_1 - \tau_h + \left(\frac{\tau_h + \tau_f}{P_K}\right) + \left(\frac{\tau_m + \tau_f}{p}\right)\left[\frac{1 - u}{P_K} + u - P_1\right]. \tag{18}$$

The characteristic function given by (17) is obtained by more general methods in Chapter 3 of this part of the report. The one-to-one correspondence between probability density and characteristic functions facilitates obtaining the unconditioned pdf of the random variable T from (17).

By the definition established in Chapter 1, Section 1.2, the reciprocal of (18) is the attrition rate for impact-lethality systems that employ the repeated single-shot, Markov firing doctrine. Special cases of (18) include

(a) <u>Equal Succeeding Round Firing Times</u> ($\tau_h = \tau_m = \tau_s$)

$$E[T] = \tau_a + \tau_1 - \tau_s + (\tau_s + \tau_f)\left[\frac{p + (1 - u) + P_K(u - P_1)}{pP_K}\right]. \tag{19}$$

(b) <u>Independent Fire</u> ($P_1 = p = u = \theta$; $\tau_h = \tau_m = \tau_s$)

$$E[T] = \tau_a + \tau_1 - \tau_s + \frac{\tau_s + \tau_f}{\theta P_K}. \tag{20}$$

(c) <u>Independent Fire, Equal Firing Times</u>
($P_1 = p = u = \theta$; $\tau_1 = \tau_h = \tau_m = \tau_s$)

$$E[T] = \tau_a + \frac{\tau_s + \tau_f}{\theta P_K} \,. \tag{21}$$

These special cases reflect the fact that the attrition rate for other impact-lethality, repeated single-shot systems are given by equation 18. For example, equation 20 may be used to determine the attrition rate for guided-missile systems. In such systems the accuracy capability of each round in a sequence is essentially the same but the timing for the first round is different from all succeeding ones.

2.2 *Burst and Mixed-Mode Firing Doctrine*

Consider next, systems that employ impact-lethality projectiles and possess the capability of burst fire. Systems of this type include the vehicle rapid-fire weapon system (VRFWS), and secondary armament on a tank. These systems can employ a number of reasonable fire doctrines such as

(a) repeated single-shot independent fire,

(b) repeated single-shot Markov fire,

(c) burst fire, and

(d) single-shot Markov fire until the first hit is obtained and then immediately switch to burst fire.

Doctrines (a) and (b) are single-shot fire doctrines, and accordingly, the attrition rate for these systems is obtained from equation 18 and special cases of it. The attrition rate for doctrine (d) is obtained by considering the single-shot and burst portions as two separate processes:

(1) single shot until the first hit is obtained, and

(2) burst fire until an additional $(z - 1)$ hits are obtained to defeat the target.

Let

n_1 = the number of rounds fired to get the first hit,

n_2 = the number of rounds fired to get $(z - 1)$ additional hits

be two random variables with expected values $E(n_1|z)$ and $E(n_2|z)$ and density functions $f_1(n_1|z)$ and $f_2(n_2|z)$, respectively. The distribution $f_1(n_1|z)$ is a special case of equation 7 (page 100), in which $z = 1$. Accordingly, $E(n_1|z)$ is given by equation 10 with $z = 1$.

$$E(n_1|z) = 1 + \frac{1 - P_1}{v} , \tag{22}$$

where

v = conditional probability of a hit following a miss but preceding the first hit

replaces the symbol p. Additionally, it is recognized that the distribution for the burst-fire phase, $f_2(n|z)$ is, except for a slight shifting of the axis, equivalent to equation 7 with the initial state probability $P_1 = 1.0$. The shifting of the distribution is due to the fact that the gunner, not waiting to observe the result of each round before firing the next one, will fire a small number of rounds while the z^{th} and last required hit is in flight to the target. Thus, from equation 10

$$E(n_2|z) = z + \frac{(z - 1)(1 - p)}{p} + c \,, \tag{23}$$

where

p = re-hitting probability (assumes the hit probability of each round in a burst is the same whether it follows a hit or a miss),

c = number of rounds fired while the round which is to become the z^{th} hit is in flight,

$= [\tau_f/\tau_b]$.[1]

[1][x] is read as the maximum integer in x. The symbol τ_b is is defined on page 111.

The total number of rounds fired to defeat the target is

$$n = n_1 + n_2 - 1, \tag{24}$$

where the minus one accounts for the fact that the first hit was counted in both processes. Since the expected value is a linear operator

$$E(n|z) = E(n_1|z) + E(n_2|z) - 1$$

$$= z + \frac{(1 - P_1)}{v} + \frac{(z - 1)(1 - p)}{p} + c \; . \tag{25}$$

Define

T_1 = time required to obtain the first hit,

T_2 = time required to obtain $(z - 1)$ additional hits,

T = time required to defeat the target (obtain a total of z hits).

Analogous to the development of equation 11,

$$T_1 = \tau_a + (\tau_1 + \tau_f) + (\tau_m + \tau_f)(n_1 - 1) \tag{26}$$

and

$$T_2 = \begin{cases} 0 & \text{for } z = 1 \\ \tau_h + \tau_f + (z - 2)\tau_b + [(n_2 - c) - z]\tau_b & \text{for } z > 1 \end{cases}$$

$$= \begin{cases} 0 & \text{for } z = 1 \\ \tau_h + \tau_f + (n_2 - c - 2)\tau_b & \text{for } z > 1, \end{cases} \tag{27}$$

where

τ_h = time to fire the first round in the burst process after obtaining the first hit in the single-shot process,

τ_b = average time between rounds during the burst firing mode. The averaging is performed over the time between individual rounds within a burst and the required cooling time between bursts.

Equation 27 is obtained by the following rationale. The gunner senses the hit and fires the first burst-mode round in τ_h seconds. That round arrives at the target τ_f seconds later. All subsequent rounds arrive in a string at the target in intervals of τ_b seconds. Excluding the c rounds fired after the round which results in the z^{th} hit (since these additional rounds do not affect the time to achieve z hits or the time to defeat the target), rounds are fired after (z - 2) hits after

the first and $[(n_2 - c) - z]$ misses. Thus, the associated expected values are

$$E(T_1|z) = \tau_a + \tau_1 - \tau_m + (\tau_m + \tau_f)E(n_1|z) \tag{28}$$

and

$$E(T_2|z) = \begin{cases} 0 & \text{for } z = 1 \\ \tau_h + \tau_f - (c + 2)\tau_b + \tau_b E(n_2|z) & \text{for } z > 1. \end{cases} \tag{29}$$

Noting that the overlap of one round between the two firing processes does not exist in the firing times

$$T = T_1 + T_2 \tag{30}$$

and

$$\begin{aligned} E(T|z) &= E(T_1|z) + E(T_2|z) \\ &= \begin{cases} c_3 + c_4 E(n_1|z) & z = 1 \\ c_3 + c_4 E(n_1|z) + c_5 + \tau_b E(n_2|z) & z > 1, \end{cases} \end{aligned} \tag{31}$$

where

$$c_3 = \tau_a + \tau_1 - \tau_m$$

$$c_4 = \tau_m + \tau_f$$

$$c_5 = \tau_h + \tau_f - (c + 2)\tau_b.$$

Removing the conditioning on the lethality variable z by

$$E(T) = \sum_{z=1}^{\infty} E(T|z)p(z) \tag{32}$$

and employing (22) and (23)

$$E[T] = \tau_a + \tau_1 + \tau_f + (\tau_m + \tau_f)\left(\frac{1 - P_1}{v}\right) + (1 - P_K)\left[\tau_h + \tau_f + \frac{\tau_b}{pP_K}(1 - pP_K)\right] . \tag{33}$$

The reciprocal of equation 33 is the attrition rate for a weapon system that employs mixed single-shot and burst-fire doctrines, and impact-lethality projectiles.

Doctrine (c), the pure burst firing mode, may be viewed as a special case of the mixed firing doctrine in which

(a) the time to fire every round except the first is τ_b,

(b) the probability $v = p$ = re-hitting probability, and

(c) only the flight time of one round need be considered.

These differences reduce equation 33 to

$$E[T] = \tau_a + \tau_1 + \tau_f - \tau_b + \tau_b\left[\frac{1 - P_K(P_1 - p)}{pP_K}\right] . \tag{34}$$

If all rounds in the burst, including the first, are independently fired, ($P_1 = p$), equation 34 reduces to

$$E[T] = \tau_a + \tau_i + \tau_f + \tau_b\left[\frac{1 - pP_K}{pP_K}\right]. \qquad (35)$$

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Appendix B, 2, 1

NUMBER OF WAYS THAT K GROUPS OF HITS CAN SUM TO Z

Seth Bonder

We seek to prove that the number of ways that k groups of hits can sum to z is $\binom{z-1}{k-1}$. Let the k groups of hits be represented by k + 1 bars. Consider initially, the problem of dropping z hits into the k groups as shown below with an x representing a hit.

```
        |xx| x |   |xxx| .....|   |xx|   .
group     1   2   3   4       k-1   k
```

The first and last bars are fixed. Therefore, this problem is one of determining the number of ways that, from z + k - 1 items (hits and bars), you can draw z hits. Equivalently, the number of ways that z hits can be arranged in z + k - 1 items, where z and k - 1 of these are different, is

$$\binom{z+k-1}{z} = \binom{z+k-1}{k-1} . \qquad (1)$$

This problem permits groups to be empty. For situations with at least one hit in each group start by dropping one hit in each, i.e., subtract k from z in (1). This produces the desired result that the number of ways that k groups of hits can sum to z is $\binom{z-1}{k-1}$.

Appendix B, 2, 2

CHARACTERISTIC FUNCTION FOR f(N|z)

Seth Bonder

By definition, the characteristic function

$$\phi_{N|z}(s) = E\left[e^{isN}\right]$$

$$= \sum_{N=0}^{\infty} e^{isN} f(N|z) \, , \qquad (1)$$

where s is a dummy variable and $i = \sqrt{-1}$. Substituting f(N|z) from the main text into equation (1),

$$\phi_{N|z}(s) = A + B \, , \qquad (2)$$

where

$$A = P_1 \left\{ e^{isz} u^{z-1} + \sum_{N=z+1}^{\infty} \sum_{k=2}^{z} e^{isN} \binom{z-1}{k-1} u^{z-k} (1-u)^{k-1} p^{k-1} \binom{N-z-1}{k-2} q^{N-z-k+1} \right\} \qquad (3)$$

and

$$B = Q_1 \sum_{N=z+1}^{\infty} \sum_{k=1}^{z} e^{isN} \binom{z-1}{k-1} u^{z-k} (1-u)^{k-1} p^{k} \binom{N-z-1}{k-1} q^{N-z-k} \, . \qquad (4)$$

Reversing the order of summation and expanding (3)

$$A = P_1 \Bigg\{ e^{isz} u^{z-1}$$

$$+\binom{z-1}{1} u^{z-2} (1-u)p \sum_{N=z+1}^{\infty} e^{isN} \binom{N-z-1}{0} q^{N-z-1}$$

$$+\binom{z-1}{2} u^{z-3} (1-u)^2 p^2 \sum_{N=z+1}^{\infty} e^{isN} \binom{N-z-1}{1} q^{N-z-2}$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\binom{z-1}{z-1} (1-u)^{z-1} p^{z-1} \sum_{N=z+1}^{\infty} e^{isN} \binom{N-z-1}{z-2} q^{N-2z+1} \Bigg\}. \tag{5}$$

By letting $y = (k-2)$, we note that the sum in the k^{th} term of equation 5 may be written

$$S_k = \sum_{N=z+1}^{\infty} e^{isN} \binom{N-z-1}{y} q^{N-z-y-1}$$

or

$$S_k = \sum_{N=z+1+y}^{\infty} e^{isN} \binom{N-z-1}{y} q^{N-z-y-1} \tag{6}$$

since $\binom{a}{b} = 0$ whenever $b > a$. Expanding (6) and recalling that

$\binom{a+b}{a} = \binom{a+b}{b}$,

$$S_k = e^{is(z+1+y)}\left[1+\binom{y+1}{1}\left(qe^{is}\right)+\binom{y+2}{2}\left(qe^{is}\right)^2+\ldots+\ldots\right] \quad (7)$$

Since

$$|qe^{is}| = |q|\ |e^{is}| = q < 1,$$

the series in the bracket of equation 7 is a binomial series of the form $(1-x)^{-n}$. Accordingly, equation 7 may be written

$$S_k = \frac{e^{is(z+1+y)}}{(1-qe^{is})^{y+1}}$$

$$= \frac{e^{is(z+k-1)}}{(1-qe^{is})^{k-1}} \quad . \qquad (8)$$

Substituting equation 8 into equation 5, term by term,

$$A = P_1 \Big\{ e^{isz}\ u^{z-1}$$

$$+\binom{z-1}{1} u^{z-2}(1-u)p\, e^{is(z+1)}(1-qe^{is})^{-1}$$

$$+\binom{z-1}{2} u^{z-3}(1-u)^2p^2e^{is(z+2)}(1-q^{is})^{-2}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\binom{z-1}{z-1} (1-u)^{z-1}p^{z-1}e^{is(2z-1)}(1-qe^{is})^{-(z-1)}$$

$$= e^{isz}P_1 \sum_{y=0}^{z-1} \binom{z-1}{y} u^{z-1-y} \left[\frac{(1-u)pe^{is}}{1-qe^{is}}\right]^y \tag{9}$$

and

$$A = e^{isz}P_1 \left[u + \frac{(1-u)pe^{is}}{1-qe^{is}} \right]^{z-1} \tag{10}$$

since the sum in equation 9 is a binomial expansion.

By a derivation analogous to equations 5 through 10, it can be shown that

$$B = \frac{e^{is(z+1)}Q_1p}{(1-qe^{is})} \left[u + \frac{(1-u)\, pe^{is}}{1-qe^{is}} \right]^{z-1} . \tag{11}$$

Substituting equation 10 and equation 11 into equation 2. the characteristic function

$$\phi_{N|z}(s) = e^{isz}\left[P_1 + \frac{Q_1 pe^{is}}{1-qe^{is}}\right]\left[u + \frac{(1-u)pe^{is}}{1-qe^{is}}\right]^{z-1} . \tag{12}$$

Chapter 3

IMPACT-LETHALITY SYSTEMS, REPEATED SINGLE-SHOT FIRE DOCTRINE, TRANSFORM APPROACH

Stephen Kimbleton

The previous chapter presented methods of developing attrition-rate prediction models for impact-lethality systems. In this chapter we present an alternate approach to developing the time-to-kill probability distribution for systems of this type. This method, based on the use of Laplace transform techniques, provides another viewpoint of the process and can be readily employed to model systems in which the probabilistic character of the process timing (acquisition, firing, etc.) is significant.

Consider the Markov firing doctrine described in Section 2.1. Although the individual sequence of hits and misses forms a Markov chain, there is a related sequence of independent, identically distributed random vartibles which is more useful in the present development. For an irreducible, positive, recurrent Markov chain, the number of transitions $S_1, S_2, \ldots$ between entries into a given state forms a sequence of independent random variables and, after the first entrance, the random variables are also identically distributed (Parzen, 1962, p.266). Employing the same notation used in Chapter 2, we observe that

$$P[S_1 = 1] = p_1$$

and for $r \geq 2$,

$$P[S_1 = r] = (1 - P_1)(1 - p)^{r-2}p \ .$$

Similarly, for $j \geq 2$,

$$P[S_j = 1] = u \ ,$$

and for $r \geq 2$,

$$P[S_j = r] = (1 - u)(1 - p)^{r-2}p \ .$$

Using these observations and proceeding via straightforward calculations, it is easy to show that

$$\begin{aligned} E[S_1] &= 1 + \frac{1 - P_1}{p} \ , \\ E[S_j] &= 1 + \frac{1 - u}{p} \ , \end{aligned} \tag{1}$$

$$\begin{aligned} VAR[S_1] &= P_1 + (1 - P_1)[2p^{-2} + p^{-1} + 1] - E[S_1{}^2] \ , \\ VAR[S_j] &= u + (1 - u)[2p^{-2} + p^{-1} + 1] - E[S_j{}^2] \end{aligned} \tag{2}$$

for $j \geq 2$,

and

$$M_{S_1}(\theta) = P_1 e^{-\theta} + \frac{Q_1 p\ e^{-2\theta}}{1 - qe^{-\theta}} \tag{3}$$

$$M_{S_j}(\theta) = ue^{-\theta} + \frac{p(1 - u)e^{-2\theta}}{1 - qe^{-\theta}} \quad ,$$

where $Q_1 = 1 - P_1$, $q = 1 - p$, and $M_X(\theta)$ is the Laplace-Stieltjes transform of the random variable X evaluated at θ. Using the preceding results, the central limit theory of renewal theory may be used to determine the number of rounds that must be fired to obtain j hits for j reasonably large. For j small, the exact distribution of the number of rounds may be obtained by brute force calculation while bounds on this distribution may be obtained through the use of Tchebycheff's inequality.

If N denotes the round on which the kill was obtained and z denoted the number of the hit on which the kill was obtained, it is seen that

$$N = S_1 + \dots + S_z \stackrel{\text{def}}{=} \sum_{j=1}^{z} S_j \ .$$

As the random variables z and S_j may be assumed independent for $j \geq 1$, it follows from a well-known theorem on random sums of random variables that (Feller, 1968, p. 287)

$$M_N(\theta) = M_{S_1}(\theta) G_{z-1}(M_S(\theta)) , \tag{4}$$

where $G_{z-1}(y)$ is the probability generating function of the random variable z - 1 with dummy variable y, and S is a random variable having the distribution of the random variables S_j for $j \geq 2$. Since the conditional kill probability, P_K, is constant for the specific case under consideration, z - 1 is geometrically distributed over the integers ≥ 0 and

$$G_{z-1}(y) = \frac{P_K}{1 - (1 - P_K)z} .$$

Observe that until this point in the argument the explicit form of z - 1 has not been used, and, indeed, (4) is valid as long as z - 1 is a non-negative integer-valued random variable. Substituting in (4),

$$M_N(\theta) = \frac{P_K M_{S_1}(\theta)}{1 - (1 - P_K)M_S(\theta)} , \tag{5}$$

and upon substituting (3),

$$M_N(\theta) = \frac{P_K P_1 e^{-\theta} + [P_K Q - P_K Q q - P_K P_1 q] e^{-2\theta}}{1 - [q + u(1 - P_K)]e^{-\theta} + [uq(1 - P_K) + p(1 - u)]e^{-2\theta}} \quad . \tag{6}$$

Hence,

$$E[N] = E[S_1] + \left(\frac{1 - P_K}{P_K}\right) E[S] \tag{7}$$

$$\begin{aligned} VAR[N] = E[S_1{}^2] &+ \left(\frac{1 - P_K}{P_K}\right) E[S^2] \\ &+ \left(\frac{1 - P_K}{P_K}\right) E[S_1]E[S] \\ &+ \left(\frac{1 - P_K}{P_K}\right)^2 E^2[S] - E^2[N] \ . \end{aligned} \tag{8}$$

Since $E[S_1]$, $E[S]$, $E[S_1{}^2]$, $E[S^2]$ have or can be expressed in terms of P_K, P_1, u, p, it follows that E[N] and VAR[N] may be so expressed by direct substitution.

In general, it is difficult to obtain the underlying probability distribution given its probability generating function. However, (6) is of the form

$$\frac{B_1 e^{-\theta} + C_1 e^{-2\theta}}{A_2 + B_2 e^{-\theta} + C_2 e^{-2\theta}} ,$$

and since this is the Laplace-Stieltjes transform of a positive integer-valued random variable, it follows that this expression also has the form

$$\sum_{n=1}^{\infty} p_n e^{-n\theta} ,$$

where $p_n = P[N = n]$. That is,

$$B_1 e^{-\theta} + C_1 e^{-2\theta} = (A_2 + B_2 e^{-\theta} + C_2 e^{-2\theta}) \sum_{n=1}^{\infty} p_n e^{-n\theta} .$$

Upon equating coefficients, and observing that $A_2 = 1$, one obtains

$$p_1 = B_1$$

$$p_2 = C_2 - B_1 B_2$$

and for $n \geq 3$, p_n satisfies the difference equation

$$p_n + B_2 p_{n-1} + C_2 p_{n-2} = 0 \ .$$

By a well-known theorem on homogeneous second-order difference equations (Goldberg, 1958, p. 141), it follows that if λ_1, λ_2 are real roots of the quadratic equation $\lambda^2 + B_2\lambda + C_2 = 0$, then

$$p_n = \beta_1\lambda_1^n + \beta_2\lambda_2^n \quad ,$$

where β_1 and β_2 are uniquely determined by the requirement that they satisfy

$$p_1 = \beta_1\lambda_1 + \beta_2\lambda_2 \ ,$$

$$p_2 = \beta_1\lambda_1^2 + \beta_2\lambda_2^2 \ .$$

If, however, λ_1, λ_2 are conjugate complex roots, then

$$p_n = \gamma r^n \cos(n\theta + \beta) \ ,$$

where λ_1 and λ_2 have the form $r(\cos\theta \pm i \sin\theta)$ and γ, β are, for $\gamma > 0$ and $0 \leq \beta < 2\pi$, uniquely determined by the requirement

$$p_1 = \gamma r \cos(\theta + \beta) ,$$

$$p_2 = \gamma r^2 \cos(2\theta + \beta).$$

Finally, if $\lambda_1 = \lambda_2 = \lambda$, then

$$p_n = (\beta_1 + \beta_2 n)\lambda^n ,$$

where β_1, β_2 are again determined from p_1, p_2. Inspection of (6) reveals that all of the preceding cases are possible. It is relatively easy to solve the above difference equations through the use of standard computational methods.

Consider next the times given in Chapter 2 as

τ_a = time to acquire target,

τ_1 = time to fire first round,

τ_h = time to fire a round given the preceding round was a hit,

τ_m = time to fire a round given the preceding round was a miss,

τ_f = projectile flight time.

We shall first assume the τ's are constants and then give a brief discussion of the modifications necessary if they are instead assumed to be random variables. For deterministic τ's, it is convenient to assume that each τ is a multiple of some fixed constant, e.g., a second or millisecond. In practice, of course, this condition is always trivially satisfied.

By virtue of the preceding assumptions, if X_1 is the random variable giving the time to the first hit and for $j \geq 2$, X_j denotes the time between the $(j - 1)^{st}$ and the j^{th} hit, then the sequence of random variables $\{X_j; j \geq 1\}$ is independent and the random variables $\{X_j; j \geq 2\}$ are identically distributed. It is also apparent that X_1 assumes only values of the form $\tau_a + \tau_1 + (r - 1)\tau_m + r\tau_f$ for $r \geq 1$, while X_j assumes only values of the form $\tau_h + (r - 1)\tau_m + r\tau_f$ for $j \geq 2$, $r \geq 1$. Indeed, we see that for $r \geq 1$

$$P[X_1 = \tau_a + \tau_1 + (r - 1)\tau_m + r\tau_f] = P[S_1 = r] ,$$

$$P[X_j = \tau_h + (r - 1)\tau_m + r\tau_f] = P[S_j = r] .$$

However, this implies that the Laplace-Stieltjes transform of X_1 and X_j are given by

$$M_{X_1}(\theta) = \exp\{-(\tau_a + \tau_1 - \tau_m)\theta\}M_{S_1}((\tau_m + \tau_f)\theta) ,$$

$$M_{X_j}(\theta) = \exp\{-(\tau_h - \tau_m)\theta\}M_{S_j}((\tau_m + \tau_f)\theta) . \tag{9}$$

If T denotes the time at which the target is destroyed or killed, then it follows that

$$T = X_1 + \ldots + X_z .$$

Employing similar arguments to those used in developing the transform $M_N(\theta)$, we have

$$M_T(\theta) = \frac{P_K M_{X_1}(\theta)}{1 - (1 - P_K)M_X(\theta)} , \tag{10}$$

where X is a random variable having the distribution of the random variables X_j for $j \geq 2$. Using (3), and (9), $M_T(\theta)$ may be expressed in terms of the basic parameters P_1, u, p, and P_K, resulting in an expression equivalent to equation 17 of Chapter 2. Either by differentiating the appropriate Laplace transform or by observing that X_j is a linear transformation of S_j for $j \geq 1$, it can be shown that

$$E[X_1] = \tau_a + \tau_1 + (E[S_1] - 1)\tau_m + E[S_1]\tau_f \; ,$$
$$E[X] = \tau_h + (E[S] - 1)\tau_m + E[S]\tau_f \; , \tag{11}$$

$$E[T] = E[X_1] + \left(\frac{1 - P_K}{P_K}\right) E[X] \; , \tag{12}$$

$$\begin{aligned} VAR[T] = {} & E[X_1{}^2] + \left(\frac{1 - P_K}{P_K}\right) E[X^2] \\ & + 2\left(\frac{1 - P_K}{P_K}\right) E[X_1]E[X] \\ & + 2\left(\frac{1 - P_K}{P_K}\right)^2 E^2[X] - E^2[T] \; . \end{aligned} \tag{13}$$

The reciprocal of (12) is the attrition rate for impact-lethality systems which employ the single-shot, Markov firing doctrine. This result was obtained in Chapter 2; however, the methods described in this chapter have a generality that can more readily be employed to model other weapon systems. Some possible extensions and benefits of this approach are listed below:

1. In obtaining the transform $M_T(\theta)$, it was implicitly assumed that the distribution of z, the number of hits required to obtain a kill, is geometric. However, as long as z is a positive

integer valued random variable the analogue of (4) will hold, i.e.,

$$M_T(\theta) = M_{X_1}(\theta) G_{z-1}(M_X(\theta)) \ .$$

2. In the preceding discussion, the τ's have been assumed to be constants. However, if the τ's are assumed to be non-negative independent random variables, the associated random variables X_j will be independent for $j \geq 1$ and identically distributed for $j \geq 2$. It follows that in this case also, expressions for the Laplace-Stieltjes transform of the time to kill may be obtained. Further, using recently developed techniques for inversion of Laplace transforms (Dubner and Abate, 1968), the exact probability distribution corresponding to (6) or (10) may be obtained.

3. Throughout our discussion we have assumed that a target which is being fired upon is, at the end of any given round, either unimpaired or destroyed. Although this is a reasonable assumption for some categories of weapons and targets, in many cases of interest there will be a number of the intermediate states of destruction of the target. At the cost of more involved computations, it is possible to extend the preceding

analysis to cover these cases. Thus, assume the various states of the target are labeled from 0 to N, state 0 corresponding to an unimpaired target and state N corresponding to a totally destroyed target.

In general, a target need not pass through all the intermediate states before being destroyed. Indeed, given a target is in state i, there may well be another state j corresponding to a greater degree of destruction of the target, and yet state j may be effectively unreachable from state i. To see this, consider the following simplified version of some results discussed by Goulet (1963). An enemy tank is assumed to be in one of four states: undamaged, mobility destroyed, firepower destroyed, or completely destroyed. (We assume that complete destruction corresponds to the destruction of both firepower and mobility.) Labeling these states from 0 to 3, respectively, it follows that if we are in state 1, state 2 may not be reached. Indeed, if we are in state 1 and the firepower capability is destroyed, it follows from our hypotheses that the state of the tank is 3. It is also of interest to note that even though a tank in state 2 would usually be regarded as having suffered more destruction than a tank in state 1, nevertheless, state 2 cannot be reached from state 1.

Assume that a tank is currently in state i and the next succeeding state of the tank is j. Then by applying the methods described in this chapter, the Laplace-Stieltjes transform of the time or number of rounds to go from i to j may be obtained. Let the sequence of successive states of destruction of a tank be $0 \to i_1 \to i_2 \to \ldots \to i_k \to N$. Then the transform of the time or number of rounds needed to go from state i_j to i_{j+1} can be obtained for $0 \leq j \leq k - 1$, and the product of the transforms then gives the transform of the time or number of rounds needed to go from state 0 to N along this particular path. The sum of these transforms over all possible paths weighted by the probability of each path then gives the (unconditioned) transform of the time or number of rounds needed to go from state 0 to state N. The number of summands would appear to be very large if the number of intermediate states of destruction is large. However, in practice, the states are usually labeled so that if we are currently in state i, then only states j with $j \geq i$ may be reached. Thus, the ultimate computational feasibility of this method depends on both the magnitude of N and the number of possible paths from 0 to N.

4. The approach used in this chapter conceptually reduces the difficulty of testing the attrition-rate models against

experimental firing data. The initial problem of drawing inferences on the parameters of a Markov chain (a difficult task) has been replaced by the significantly simpler problem of drawing inferences as to the independence and identical distribution of sequences of random variables.

3.1 References

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Chapter 4

SEMI-MARKOV ANALYSIS

Robert Farrell

In Chapters 2 and 3, we described two methods of obtaining time-to-kill probability distributions for impact-lethality, repeated single-shot weapons. The attrition rates of these weapons are obtained as the reciprocal of the mean time to kill. This chapter treats a general method of developing such attrition rates without analyzing the complete distribution of the time to kill. The approach taken in this development is based on the theory of semi-Markov or Markov-renewal processes, and is a generalization of the methods in Barfoot (1969).

Basically, we analyze the process in which a weapon fires at a target until he decides to cease fire on it, fires at a second target until he decides to cease fire on it, etc. This process is analyzed by subdividing the period of fire on a single target into intervals corresponding to differences in the behavior or state of the firing weapon system.

This technique may be used to determine the expected time to kill in any firing process with a set of distinguishable states $S_1,\ldots,S_N$ (e.g., first round fired, round fired after a preceding hit, etc.) as long as

(a) the process makes transitions at distinct points in time (shell arrivals in the example);

(b) the probability of transition to S_j, given one is in S_i, is p_{ij} which does not depend on knowledge of any history of the process;

(c) given an entry into S_i and the next transition from S_i to S_j, the length of time in the interval from entry to exit is a random variable distributed as F_{ij}, which may depend on the states S_i and S_j but is not influenced by further knowledge of the process history. This random time interval has a finite mean, m_{ij};

(d) the process starts in S_1 (finished with last engagement, starting new one) and terminates with an entry to S_1; and

(e) every state has some probability of eventually occurring.

In essence, the technique is applicable for any continued firing process which may be modeled as a semi-Markov process.

We first define

$$m_i = \sum_{j=1}^{N} p_{ij} m_{ij}$$

and f_i, the Markov-chain steady-state frequencies,[1] as the solution of

$$f_i = \sum_{j=1}^{N} f_j p_{ji} \;, \quad \sum_{i=1}^{N} f_i = 1 \;.$$

[1]The meaning and properties of the steady-state frequencies are discussed in any book on stochastic processes or Markov chains. See, for instance, Parzen (1962), Kemeny and Snell (1960), or Karlin (1966).

Then from an elementary theorem of Markov renewal theory,[2] we know that

$$E(T) = f_1^{-1} \sum_{j=1}^{N} f_j m_j .$$

As an example of the use of this method, let us consider a generalized version of the "Markov fire" case treated in Chapter 2. Let

S_1 = state preceding first round at new target after termination of an engagement,

S_2 = state after a hit (which did not kill) on current target,

S_3 = state after a miss (which did not kill) on current target,

u = probability of a hit after a preceding hit,

p = probability of a hit after a preceding miss,

P_1 = probability of a hit on the first round,

H_H = probability that a hit after a hit kills the target,

H_M = probability that a hit after a miss kills the target,

H_1 = probability that a hit on first round kills the target,

M_H = probability that a miss after a hit kills the target,

M_M = probability that a miss after a miss kills the target,

M_1 = probability that a miss on the first round kills the target.

Then we have

---

[2]This <u>is</u> theorem 5.16 in Ross (1970) and theorem 6.12 in Cinlar (1969).

$$
\begin{aligned}
p_{11} &= P_1H_1 + (1 - P_1)M_1, \\
p_{12} &= P_1(1 - H_1), \\
p_{13} &= (1 - P_1)(1 - M_1), \\
p_{21} &= uH_H + (1 - u)M_H, \\
p_{22} &= u(1 - H_H), \\
p_{23} &= (1 - u)(1 - M_H), \\
p_{31} &= pH_M + (1 - p)M_M, \\
p_{32} &= p(1 - H_M), \\
p_{33} &= (1 - p)(1 - M_M).
\end{aligned}
$$

We will assume the distributions F_{ij} or the composite $\sum_{j=1}^{N} p_{ij}F_{ij}$ are available, and that the m_i have been determined.[1] Now, solving the steady-state equation gives[2]

$$
\begin{aligned}
f_1 &= (1 + a_2 + a_3)^{-1} \\
f_2 &= a_2/(1 + a_2 + a_3) \\
f_3 &= a_3/(1 + a_2 + a_3)
\end{aligned}
$$

where

$$
\begin{aligned}
a_2 &= (p_{32}(1 - p_{11}) + p_{12}p_{31})/(p_{21}p_{32} + (1 - p_{22})p_{31}) \\
a_3 &= ((1 - p_{11})(1 - p_{22}) - p_{12}p_{21})/((1 - p_{22})p_{31} + p_{21}p_{32}) .
\end{aligned}
$$

And finally ,

$$E(T) = m_1 + a_2m_2 + a_3m_3 . \tag{1}$$

[1]Any data which determine the m_i are adequate; no particular forms are required.

[2]There are many alternative forms for this solution. This may not be the most appropriate for computational purposes.

It may be noted that although independent data entries $(u,p,P_1,H_H,H_M,H_1,M_H,M_M,M_1,m_1,m_2,m_3)$ are required to describe the entire process, only 5 dimensions of freedom exist in the E(T) expression (a_2,a_3,m_1,m_2,m_3). Further, a_2 and a_3 may be determined from 6, not 9, expressions $(p_{11},p_{12},p_{21},p_{22},p_{31},p_{32})$. Thus , a data-generation and data-handling savings may result if some of these compressed forms could be obtained to replace the 12 (or more,) if the F_{ij} or m_{ij} are considered original dimensions.

It is clear that (1) could be rewritten to give an expression for E(T) in terms of the fundamental process parameters by using the expressions for m_1, m_2, and m_3. The present form is slightly more convenient for computational purposes, however.

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Chapter 5

AREA-LETHALITY SYSTEMS

Robert Gruhl and Robert Farrell

This chapter presents the development of a model to predict the attrition rate for one or more weapons classified as area-lethality systems. Systems of this type usually fire into an area without knowledge of exact target locations and destroy targets via fragmentation or some other area-lethality mechanism. A field artillery battery is an example of this type of system and the problem is to determine the time rate of destroying an area target by the simultaneous and sequential delivery of multiple weapons in the battery.

The attrition-rate model developed in this chapter employs results of the multivolley target coverage analysis conducted by Hess (1968). Integral to his analysis (and thus, the formulas developed herein) are some specific "target coverage functions" and "damage functions"; however, the approach used to develop the attrition-rate model can readily consider other coverage and damage functions.

Because of the reliance on target coverage methodology and the use of Hess's specific assumptions and results, these are briefly reviewed in Section 5.1. The attrition-rate model is developed in Section 5.2. The effect of changing target

posture during a firing interval is considered in Section 5.3. Section 5.4 contains a discussion of modifications to account for possible nonhomogeneous damage levels within the target area.

5.1 Multivolley Target Coverage

The target coverage problem concerns methods for determining the damage to targets inflicted by the delivery of one or more indirect-fire weapons. Usually the coverage problem is used to denote the one-shot problem, while the multivolley problem denotes more than one shot. A volley is the number of rounds fired from a group of identical weapons (four to eight in firing battery). A comprehensive bibliography of coverage problem literature can be found in Guenther and Terragno (1964).

The multivolley coverage analysis used in the development of the attrition-rate model is given by Hess (1968). Except for the damage pattern assumption, Hess's model for the expected fraction of damage to the target is based on a minimum set of general assumptions. The following specific assumptions were used for model verification, application, and computational purposes:

Delivery Bias

No delivery bias exists--no aiming error, target location error, or intentional offset.

Delivery Error

Centers of impact (x,y) of the volley damage patterns are distributed about a mean center of impact $(\bar{x},\bar{y})$ according to the circular normal distribution. For convenience, we let $(\bar{x},\bar{y}) = (0,0)$, $\sigma = 1$. The delivery error is then

$$b(x,y) = (2\pi)^{-1}\exp[-(x^2 + y^2)/2] \ . \qquad (1)$$

Target

A circle with radius R_t centered at the origin. Two, mathematically equivalent, targets are considered:

a. A circular homogeneous-area target, centered at $(0,0)$ and radius R_t, and

b. A point target (ξ,η) of uniformly uncertain location in the area of radius R_t. The target density function $\underline{W}(\xi,\eta)$ is then:

$$\frac{1}{\pi R_t^2}, \text{ where } \iint_{\text{target}} \underline{W}(\xi,\eta)d\xi d\eta = 1 \ .$$

Damage Assumption

The damage pattern is a circular cookie-cutter[1] of radius R_p. Let $\underline{d}(\xi,\eta;x,y)$ be the damage function:

[1]It has been shown (Gates, 1954) that a circular coverage damage function, which is a more realistic portrayal of an actual damage pattern, yields the same results as the circular cookie-cutter if weapons are delivered with circular normal errors except for a larger delivery variance found with the circular coverage damage function. Hence, derived results are applicable to a better damage function than the cookie-cutter.

$$\underline{d}(\xi,\eta;x,y) = \begin{cases} \lambda, & (x-\xi)^2 + (y-\eta)^2 \leq R_p^2 \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

where

$\underline{d}(\xi,\eta;x,y)$ is the probability that a point target at (ξ,η) will be killed by a damage pattern with center of impact at (x,y). Damage is either all or nothing (killed or not killed)--no cumulative damage is considered.

We proceed by letting

r = distance from (ξ,η), a point target, to mean center of impact $(0,0)$ so that

$$r^2 = \xi^2 + \eta^2,$$

and

$P(R_p,r)$ = the probability that a point target (ξ,η) is covered by a damage pattern with center of impact at (x,y) [this is also the probability that a volley center of impact, subject to the circular normal distribution, will fall within a circle of radius R_p around (ξ,η)].

Then,

$$P(R_p,r) = \iint_C \frac{1}{2\pi} \exp[-(x^2 + y^2)/2]dxdy, \tag{3}$$

where C is the circle

$$(x-\xi)^2 + (y-\eta)^2 \leq R_p^2 .$$

The event that the point (ξ,η) is covered by a volley is a Bernoulli random variable. Then the probability of covering

(ξ,η) k times in v volleys is binomial:

$$g(k;v,R_p,r) = \binom{v}{k} [P(R_p,r)]^k [1 - P(R_p,r)]^{v-k} . \quad (4)$$

This is the point coverage function.

Letting

D = the event a target element is damaged,

C_k = the event a target element is covered exactly k times,

and assuming independence of damage (kill) between volleys:

$$P(D|C_k) = 1 - (1 - \lambda)^k . \quad (5)$$

The joint probability of covering a target element located at (ξ,η) exactly k times in v volleys and damaging it is

$$\begin{aligned} P_v(D \cdot C_k) &= P(D|C_k)P(C_k) \\ &= g(k;v,R_p,r)[1 - (1 - \lambda)^k] . \quad (6) \end{aligned}$$

The marginal probability of damaging an element located at (ξ,η) in v volleys is

$$P_v(D) = \sum_{k=1}^{v} P_v(D \cdot C_K)$$

$$= \sum_{k=1}^{v} g(k;v,R_p,r)[1 - (1 - \lambda)^k] \ . \qquad (7)$$

Defining

$$P(W) = \text{probability that the point target is at } (\xi,\eta)$$

$$= W(\xi,\eta) = \frac{1}{\pi R_t^2} \qquad \xi^2 + \eta^2 \leq R_t^2 \ , \qquad (8)$$

then $P_v(W \cdot D) = P(W)P_v(D)$ and the marginal probabilities of damage (kill) to a point target, $\bar{f}_v$, is

$$\bar{f}_v = \iint_{\text{target}} P_v(W \cdot D)d\xi d\eta$$

$$= \iint_{\text{target}} \frac{1}{\pi R_t^2} \sum_{k=1}^{v} g(k;v,R_p,r)[1 - (1 - \lambda)^k]d\xi d\eta$$

$$= \sum_{k=1}^{v} [1 - (1 - \lambda)^k] \iint_{\text{target}} \frac{1}{\pi R_t^2} g(k;v,R_p,r)d\xi d\eta \ . \qquad (9)$$

The target coverage function is defined as

$$G(k;v,R_p,R_t) = \iint_{\text{target}} \frac{1}{\pi R_t^2}\, g(k;v,R_p,r)\,d\xi\,d\eta \ . \tag{10}$$

$G(k;v,R_p,R_t)$ is the expected fraction of target area covered exactly k times in v volleys or the probability that a point target is covered exactly k times in v volleys by the damage pattern. Thus,

$$\bar{f}_v = \sum_{k=0}^{v} [1 - (1 - \lambda)^k] G(k;v,R_p,R_t) \tag{11}$$

is the expected fraction of damage to an area target in v volleys or the probability that a point target of uncertain location within the target area is damaged (killed). Employing the specific assumptions noted above, it can be shown that

$$\bar{f}_v = 1 - \sum_{k=0}^{v} (-1)^k \binom{n}{k} \lambda^k S_k(R_p,R_t) \ , \tag{12}$$

where

$$S_k(R_p,R_t) = \frac{2}{R_t^2} \int_0^{R_t} [P(R_p,r)]^k r\,dr \tag{13}$$

with $P(R_p,r)$ given by (3).

A large number of integrations are involved in the calculations of $\bar{f}_v$. Hess developed an approximation, $\hat{f}_v$, to $\bar{f}_v$ by replacing $G(k;v,R_p,R_t)$ with

$$Q(k;v,R_p,R_t) = \binom{v}{k}[S_1(R_p,R_t)]^k[1 - S_1(R_p,R_t)]^{v-k} , \quad (14)$$

where $S_1(R_p,R_t)$ is the probability a target element is at (ξ,η) and covered by the damage pattern, or the expected fraction of the target covered by the damage pattern *in one volley*. The resulting approximation is given by

$$\hat{f}_v = \sum_{k=0}^{v} [1 - (1 - \lambda)^k]Q(k;v,R_p,R_t)$$

$$= 1 - [1 - \lambda S_1]^v , \quad (15)$$

where $S_1(R_p,R_t)$ is denoted by S_1.

The approximation $\hat{f}_v$ is subject to large error if R_p or R_t are not small relative to the circular probable error, CEP (radius of a circle centered at the mean impact point containing 50 percent of the impact locations). A correction factor CF_v was devised by Hess which corrects, for the basic assumptions above, the approximation $\hat{f}_v$ to within 1 percent of $\bar{f}_v$. The

correction factor is given by

$$CF_v = 1 - (v - 1)\gamma e^{-(v-2)\delta} . \tag{16}$$

The parameters γ and δ are charted by Hess (1968, pp. 212-21). Thus, the corrected approximate expected fraction of damage is

$$\begin{aligned} \hat{F}_v &= \hat{f}_v CF_v \\ &= [1 - (1 - \lambda S_1)^v]CF_v . \end{aligned} \tag{17}$$

5.2 *The Attrition Rate*

$\hat{F}_v$ is dependent on the number of volleys, v. Assuming a constant firing rate, f, the corrected approximate expected fraction of area target killed as a function of time, denoted by $\phi_c(t)$, is

$$\phi_c(t) = [1 - (1 - \lambda S_1)^{ft}]CF_{ft} . \tag{18}$$

If N is the number of independent and identically distributed targets in the area at the beginning of the time

interval [0,t], the expected number[1] at t is

$$n(t) = [1 - \phi_c(t)]N \ .$$

The expected number at $(t + \tau)$ is then

$$n(t + \tau) = [1 - \phi_c(t + \tau)]N \ .$$

Then

$$\frac{dn}{dt} = \lim_{\tau \to 0} \frac{n(t + \tau) - n(t)}{\tau}$$

$$= \lim_{\tau \to 0} \left\{[1 - \phi_c(t + \tau)]N - [1 - \phi_c(t)]N\right\}$$

$$= \lim_{\tau \to 0} - N \frac{\phi_c(t + \tau) - \phi_c(t)}{\tau}$$

$$= -N\phi_c'(t) \ . \tag{19}$$

Comparing this expression with [1] of Chapter 2, Part A, for a single Red group $(J = 1)$ and only one firing Blue unit $(m = 1)$ suggests that the attrition rate for indirect-fire, area-lethality systems be taken as

$$\alpha = \phi_c'(t)N \ . \tag{20}$$

---

[1] This is based on the assumption that the probability mass function of the number of survivors is binomial with parameter $[1 - \phi(t)]$.

A useful simplification of (10) for numerical evaluation of the general combat equations is obtained if we use the uncorrected approximate expected fraction of area target killed in (20). That is, substitute

$$\phi(t) = 1 - (1 - \lambda S_1)ft \tag{21}$$

for $\phi_c(t)$. Then

$$\phi'(t) = -f(1 - \lambda S_1)^{ft} \ln(1 - \lambda S_1) \ . \tag{22}$$

But $(1 - \lambda S_1)^{ft}$ is the fraction of area target not damaged, and therefore,

$$(1 - \lambda S_1)^{ft} = \frac{n(t)}{N} \ . \tag{23}$$

Substituting (23) into (22),

$$\phi'(t) = -f\left(\frac{n(t)}{N}\right) \cdot \ln(1 - \lambda S_1) \ .$$

Then,

$$\frac{dn}{dt} = -N\left[\frac{n(t)}{N}f \cdot \ln(1 - \lambda S_1)\right]$$

$$= -n(t)f \cdot \ln(1 - \lambda S_1) \ .$$

But $-f \cdot \ln(1 - \lambda S_1) = \phi'(0)$ when $\phi(t) = [1 - (1 - \lambda S_1)^{ft}]$. Therefore, the attrition of an area target due to indirect fire from one Blue firing unit is

$$\frac{dn}{dt} = -n(t)\phi'(0)$$

and

$$\alpha_{uncorrected} = \phi'(0)n(t) \ . \tag{24}$$

This simplified form of the attrition rate should be used only when Hess's uncorrected approximation,

$$\phi(t) = 1 - (1 - \lambda S_1)^{ft} \ ,$$

is appropriate and the interpretation of (23) can be given to $(1 - \lambda S_1)^{ft}$. It is dependent on the assumption that the targets continuously uniformly distributed themselves in the target area and therefore that the probability of n(t) survivors is

binomial. In general, $\phi(t)$ is a good approximation if $R_p >> R_t$ or when $R_t < \sigma$, $R_p < \sigma$ (where R_p is the radius of the lethal effects circle, R_t is the radius of the target and σ is the standard deviation of the delivery error) or when the number of volleys, $v = ft$, is small, e.g.,

> $\hat{f}_v = \phi(t)$ is a good approximation to $\bar{f}_v$ for $v < 10$
> when $S_1 = .2183$, $\lambda = .25$, $R_p = 1$ CEP, $R_t = 2$ CEP
> (see Hess, 1968, p. 88).

Returning to the basic attrition rate (equation 20),

$$\phi_c'(t) = \frac{d}{dt}\,[1 - (1 - \lambda S_1)^{ft}][1 - (ft - 1)\gamma e^{-(ft-2)\delta}] \qquad (25)$$

$$= -f\gamma e^{-(ft-2)\delta}\Big\{1 - \delta(ft - 1) - (1 - \lambda S_1)^{ft}$$

$$\cdot\, [(ft - 1)\ln(1 - \lambda S_1) + 1 - \delta(ft - 1)]\Big\}$$

$$-f(1 - \lambda S_1)^{ft}\ln(1 - \lambda S_1)$$

after some algebraic manipulations. Employing (22) and letting

$$c'(t) = -f\gamma e^{-(ft-2)\delta}\{1 - \delta(ft - 1) - (1 - \lambda S_1)^{ft}$$

$$\cdot\, [(ft - 1)\ln(1 - \lambda S_1) + 1 - \delta(ft - 1)]\}\,, \qquad (26)$$

$$\phi_c'(t) = c'(t) + \phi'(t) \ . \tag{25}$$

Thus, the attrition rate using the corrected approximate expected fraction of damage to an area target is

$$\alpha = [c'(t) + \phi'(t)]N \ . \tag{26}$$

5.3 Different Target Postures

The basic model assumes that the target vulnerability does not change during a volley attack. However, in practice, target elements (e.g., personnel) usually respond to an attack by changing location and/or posture in order to decrease their vulnerability. In this section we consider the change in target posture (e.g., from standing to prone to being in a foxhole) following Hess's analysis and show its effect on the uncorrected attrition rate.

Let

ℓ = the number of target postures, $\ell \geq 1$,

λ_i = probability of kill given a coverage while in the i^{th} posture, $i = 1, 2, \ldots \ell$,

v_i = the corresponding number of volleys the target is in the i^{th} posture such that

$$\sum_{i=1}^{\ell} v_i = v = \text{total number of volleys.}$$

Then the uncorrected approximate probability of kill in v volleys, $\hat{f}_v$, to a point target of uncertain location within the area (or the fraction of damage to an area target) is (Hess, 1968, p. 94)

$$\hat{f}_v = 1 - \prod_{i=1}^{\ell} (1 - \lambda_i S_1)^{v_i} \tag{27}$$

or alternatively

$$\hat{f}_v = 1 - [1 - \lambda(v) S_1]^v , \tag{28}$$

where

$\lambda(v)$ = the expected probability of kill given coverage in v volleys,

$$= 1 - \prod_{i=1}^{\ell} (1 - \lambda_i)^{v_i/v} . \tag{29}$$

Let $v = ft$ in (28) and (29) and $v_i = ft_i$ in (28), where t_i is the amount of time spent in posture i and

$$\sum_{i=1}^{\ell} t_i = t .$$

Thus,

$$\phi(t) = [1 - (1 - \lambda(ft)S_1)^{ft}]$$

$$\phi'(t) = \frac{d}{dt}[-(1 - \lambda(ft)S_1)^{ft}]$$

$$= \frac{d}{dt}\left\{(1 - a(t)]^{b(t)}\right\},$$

where $a(t) = \lambda(ft)S_1$ and $b(t) = ft$. Thus,

$$\phi'(t) = [1 - a(t)]^{b(t)} \cdot \Big\{b'(t)\ln[1 - a(t)]$$

$$+ b(t)\frac{1}{1 - a(t)}[-a'(t)]\Big\} . \tag{30}$$

Evaluating the derivatives,

$$b'(t) = \frac{d}{dt}(ft) = f$$

$$a'(t) = \frac{d}{dt}[\lambda(ft)S_1] = S_1 \frac{d}{dt}\lambda(ft) ,$$

where

$$\lambda(ft) = 1 - \prod_{i=1}^{\ell} (1 - \lambda_i)^{t_i/t} .$$

Let $t_i = a_i t$, where $0 \leq a_i \leq 1$ is the fraction of the total firing time the target spends in posture i. Then,

$$\lambda(ft) = 1 - \prod_{i=1}^{\ell} (1 - \lambda_i)^{a_i t/t}$$

$$= 1 - \prod_{i=1}^{\ell} (1 - \lambda_i)^{a_i} \overset{\text{def}}{=} \lambda_a \, . \tag{31}$$

Thus, $\frac{d}{dt}\lambda(ft) = 0$, and $a'(t) = 0$. Therefore, from (30)

$$\phi'(t) = -(1 - \lambda_a S_1)^{ft} f\ln(1 - \lambda_a S_1) \, ,$$

and setting $t = 0$,

$$\phi'(0) = -f\ln(1 - \lambda_a S_1) \, , \tag{32}$$

where λ_a is given by (31). Equation 32 is used directly in (24) for computing the uncorrected attrition rate with target posture changes.

5.4 Nonhomogeneous Lethality

The attrition-rate model using the uncorrected approximate expected fraction of area target killed assumes that λ, the probability of kill given a coverage, is known for a given

target and weapon combination and is constant over a given target area (i.e., a target element is not more vulnerable in one part of the area target than another). In this section we show how the model can be extended to include varying degrees of vulnerability in the area with respect to one target type.

The basic assumptions for the extension are similar to the previous development except

Damage

Assume the damage pattern is a circular cookie-cutter of radius R_p:

$$\underline{d}(\xi,\eta;x,y) = \begin{cases} \lambda_1 & (x-\xi)^2 + (y-\eta)^2 \leq R_p^2,\ \eta \geq 0 \\ \lambda_2 & (x-\xi)^2 + (y-\eta)^2 \leq R_p^2,\ \eta < 0 \\ 0 & \text{otherwise}, \end{cases} \tag{33}$$

where (ξ,η) is the location of a target element uniformly distributed in the target area, the circle of radius R_t.

Thus, the probability of damaging the target with a single coverage is different depending on whether or not the target is in one semicircle or the other which comprise the target area. Similar to the previous development, let

D = the event a target element is damaged,

C_K = the event a target element is covered exactly K times,

A_j = the event (ξ,η) is in target area j $(j = 1,2)$,

which leads directly to

$$P(D|C_k \cdot A_1) = 1 - (1 - \lambda_1)^k \tag{34}$$

$$P(D|C_k \cdot A_2) = 1 - (1 - \lambda_2)^k . \tag{35}$$

Since the probability of C_k is given by (4) and

$$P(A_1) = P(A_2) = (2\pi R_t^2)^{-1} , \tag{36}$$

then

$$P(D \cdot C_k \cdot A_1) = (2\pi R_t^2)^{-1} g(k;v,R_p,r)[1 - (1 - \lambda_1)^k] \tag{37}$$

$$P(D \cdot C_k \cdot A_2) = (2\pi R_t^2)^{-1} g(k;v,R_p,r)[1 - (1 - \lambda_2)^k]. \tag{38}$$

Letting D_v be the damage in v volleys,

$$\begin{aligned} P(D_v \cdot A_1) &= \sum_{k=1}^{v} Pr\{D \cdot C_k \cdot A_1\} \\ &= (2\pi R_t^2)^{-1} \sum_{k=1}^{v} \left\{ \binom{v}{k} [P(R_p,r)]^k \right. \\ &\qquad \left. \cdot [1 - P(R_p,r)]^{v-k} [1 - (1 - \lambda_1)^k] \right\} \end{aligned}$$

and

$$P(D_v \cdot A_2) = \sum_{k=1}^{v} \Pr\{D \cdot C_k \cdot A_2\}$$

$$= (2\pi R_t^2)^{-1} \sum_{k=1}^{v} \left\{ \binom{v}{k} [P(R_p,r)]^k \right.$$

$$\left. \cdot [1 - P(R_p,r)]^{v-k} [1 - (1 - \lambda_2)^k] \right\} .$$

Since the events A_1 and A_2 are mutually exclusive for a single target in the total area A,

$$P(D_v \cdot A) = (2\pi R_t^2)^{-1} \sum_{k=1}^{v} \left\{ \binom{v}{k} [P(R_p,r)]^k \right.$$

$$\cdot [1 - P(R_p,r)]^{v-k}$$

$$\left. \cdot \left\{ [1 - (1 - \lambda_1)^k] + [1 - (1 - \lambda_2)^k] \right\} \right\}$$

$$= (\pi R_t^2)^{-1} \sum_{k=1}^{v} \left\{ \binom{v}{k} [P(R_p,r)]^k [1 - P(R_p,r)]^{v-k} \right.$$

$$\left. \cdot [1 - \tfrac{1}{2}(1 - \lambda_1)^k - \tfrac{1}{2}(1 - \lambda_2)^k] \right\}$$

and

$$P(D_v) = \bar{f}_v = \iint_{\text{target}} (\pi R_t^2)^{-1} \sum_{k=1}^{v} \Big\{ g(k;v,R_p,r)$$

$$\cdot \Big\{1 - \tfrac{1}{2}[(1 - \lambda_1)^k + (1 - \lambda_2)^k]\Big\}\Big\} d\xi d\eta$$

$$= \sum_{k=1}^{v} \Big\{ \Big\{[1 - \tfrac{1}{2}[(1 - \lambda_1)^k + (1 - \lambda_2)^k]\Big\}$$

$$\cdot \iint_{\text{target}} (\pi R_t^2)^{-1}\, g(k;v,R_p,r) d\xi d\eta \Big\}. \tag{39}$$

The double integral in (39) is the target coverage function $G(k;v,R_p,R_t)$ given by (10) which, as an approximation, can be replaced by $Q(k;v,R_p,R_t)$ given by (14). Analogous to the previous development this leads to the uncorrected approximate expected fraction damage $\hat{f}_v \approx \bar{f}_v$.

$$\hat{f}_v = \sum_{k=1}^{v} \binom{v}{k} S_1^k (1 - S_1)^{v-k}$$

$$- \frac{1}{2} \sum_{k=1}^{v} \binom{v}{k} [S_1(1 - \lambda_1)]^k (1 - S_1)^{v-k}$$

$$- \frac{1}{2} \sum_{k=1}^{v} \binom{v}{k} [S_1(1 - \lambda_2)]^k (1 - S_1)^{v-k}$$

$$= 1 - \frac{1}{2} [(1 - \lambda_1 S_1)^v + (1 - \lambda_2 S_1)^v] . \tag{40}$$

Equation 40 can be used directly as $\phi(t)$ to estimate the attrition rate with the uncorrected approximate expected fraction damage.

By induction, the analysis of this section can be extended to m different damage levels associated with m equal partitions of the target circle. This results in the approximation

$$\hat{f}_v = 1 - \frac{1}{m}\left[\sum_{j=1}^{m} (1 - \lambda_j S_1)^v\right]. \tag{41}$$

5.5 *References*

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PART C

HOMOGENEOUS-FORCE DIFFERENTIAL MODELS

The basic structure assumed to describe the combat activity was given in Part A by the coupled sets of differential equations

$$\frac{dn_j}{dt} = -\sum_i A_{ij}(r)m_i \qquad \text{for } j = 1,2,\ldots,J \;, \qquad [1]$$

$$\frac{dm_i}{dt} = -\sum_j B_{ji}(r)n_j \qquad \text{for } i = 1,2,\ldots,I \;. \qquad [2]$$

The preceding part of the report described methods that have been developed to predict the principal input to these equations--the attrition rate. This and the next part of the report present results of research that has been directed to obtaining solutions for the above equations, where a solution is taken to be the trajectory of surviving forces of each group during the battle as a function of basic inputs and initial numbers of forces.[1]

Ideally, it would be desirable to have the solutions in simple, closed form which would readily portray the relationship between the independent factors of the combat process and the surviving numbers of forces. This would facilitate

[1]Logistics and locations of survivors can also be determined as part of the solution, but are omitted in this discussion.

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both sensitivity analysis and determination of those independent variables which significantly contribute to combat effectiveness. Attempts to obtain such closed-form solutions have focused on simplified cases of the combat equations in order to obtain some insight into the solution procedures and problems related thereto. These simplified cases include (a) homogeneous-force battles (one group on each side) which are described in this part of the report, and (b) constant-coefficient, heterogeneous-force battles which are described in Part D.

Chapter 1 considers the case of constant attrition rates for both the Red and Blue weapons. Chapter 2 presents the solution to a special case of variable attrition rates in which their ratio is a constant. The effect of assault speed under this condition is examined in Chapter 3. Chapter 4 presents some approximation results for general variable attrition rates in homogeneous-force battles. Analog solutions for linear attrition-rate functions are presented in Chapter 5. Analytic solutions for a hypothetical fire-support situation with variable attrition rates are given in Chapter 6.

Chapter 1

CONSTANT ATTRITION-RATE MODEL

Seth Bonder

In this chapter we consider the simplest homogeneous-force battle model in which the attrition rates are constant and the intelligence coefficients are unity.[1] The constancy of the attrition rates indicates that they are neither functions of battle time nor range between weapon and target groups. Since there is only one group on each side, the allocation factor is also equal to unity for each force.

These assumptions reduce the heterogeneous-force battle equations to

$$\frac{dn}{dt} = -\alpha m \qquad (3)$$

$$\frac{dm}{dt} = -\beta n \qquad (4)$$

if the attrition rates are also not functions of the number of surviving targets and

$$\frac{dn}{dt} = -\alpha_A nm \qquad (5)$$

$$\frac{dm}{dt} = -\beta_A mn \qquad (6)$$

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[1] All research presented in this report has considered unity intelligence coefficients.

when both sides employ area-lethality weapons. The attrition rates in (5) and (6), ($\alpha_A n$) and ($\beta_A m$), reflect the dependency of the uncorrected area-lethality attrition rates developed in [B,5.0] on the number of surviving targets where, notationally, α_A and β_A are given by $\phi'(0)$ of that chapter.[1]

Equations 3 to 6 are the classical combat formulations of F. W. Lanchester (1916). Equations 3 and 4 comprise the more familiar "modern combat" description in which it is assumed that combat takes place at close quarters such that each unit may take any enemy unit under fire and, having killed that enemy unit, shifts its fire to another enemy unit. Combatants whose weapon systems have attrition rates classified as impact lethality (see [B, 2.0]), *and are constant throughout the battle,* would be consistent with this formulation. This description additionally assumes that units on either side are within weapon range of all enemy units and that fire is distributed uniformly over remaining units.

The solution of equations 3 and 4 with the time variable removed--called the state solution--is obtained by dividing (3) by (4), integrating, and employing the initial force size conditions that, at $t = 0$, $n = N$ and $m = M$. This

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[1]The attrition rates for area lethality systems are the only ones developed to date which are state dependent. Accordingly the battle description given by (5) and (6) is the only state-dependent attrition-rate case examined in this report. Other hypothesized state-dependent descriptions are summarized by Dolanský (1964).

leads to the result that

$$\alpha(M^2 - m^2) = \beta(N^2 - n^2) \quad , \qquad (7)$$

which is invariant throughout the battle. Thus, for any specified number of surviving Red units, we can determine the associated number of surviving Blue units. For example, if the Red force is annihilated (n = 0), then

$$m^2 = \alpha M^2 - \beta N^2 \quad , \qquad (8)$$

which indicates that Blue will have some surviving units if

$$\alpha M^2 > \beta N^2 \quad . \qquad (9)$$

Inequality (9) implies that Blue will win the battle, *if winning is taken to be annihilation of the opposing force.* The condition

$$\alpha M^2 = \beta N^2 \qquad (10)$$

implies a draw (Red and Blue forces approach zero simultaneously). Lanchester called this condition an equality of *fighting strengths* and, since it is proportional to the square of the force size, has been given the familiar name

"Lanchester's square law."[1] This suggests that there exists a definite advantage in concentrating forces. If the Blue force has a weapon whose attrition rate is four times greater than the Red force weapons, the Red force will require only twice the initial number of forces to have equal potential of annihilating the Blue force.

The time solution[2] of this simplified description of combat is well known and readily obtained by substituting (3) into the derivative of (4) and solving the resulting second-order, constant coefficient, differential equation under the initial conditions that $n = N$ and $m = M$ at $t = 0$ producing

$$n = N \cosh (\sqrt{\alpha\beta}t) - \sqrt{\alpha/\beta}\, M \sinh (\sqrt{\alpha\beta}t) \qquad (11)$$

and

$$m = M \cosh (\sqrt{\alpha\beta}t) - \sqrt{\beta/\alpha}\, N \sinh (\sqrt{\alpha\beta}t) \quad . \qquad (12)$$

It is also a straightforward task to determine the time

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[1]Weiss (1962) notes that Lanchester's square law was apparently anticipated by Rear Admiral Bradley A. Fiske as early as 1905. Fiske stated that (Robison, 1942): "The decrease in offensive power of a weaker fleet fighting a stronger is geometrical, instead of arithmetical, and that there is a continually increasing difference between the powers of two fleets as an action progresses which favors the stronger fleet." This is the effect of concentration described by Lanchester's equations. Although Fiske qualitatively described this phenomena, Lanchester was the first to formalize it in quantitative terms.

[2]Number of surviving Red and Blue units as a function of battle time.

(τ_o^i) required for the i^{th} side to be completely annihilated as the min $\left[\tau_o^n, \tau_o^m\right]$, where

$$\tau_o^n = \frac{1}{\sqrt{\alpha\beta}} \tanh^{-1}\left(\frac{N}{M}\sqrt{\beta/\alpha}\right) \tag{13}$$

$$\tau_o^m = \frac{1}{\sqrt{\alpha\beta}} \tanh^{-1}\left(\frac{M}{N}\sqrt{\alpha/\beta}\right) . \tag{14}$$

These are derived from equations 11 and 12 by setting the left-hand side equal to zero and solving for the appropriate time.

Equations 5 and 6 contain state-dependent attrition rates derived in [B, 5.0] for weapon systems that use area-lethality mechanisms. The implied assumptions are (a) the targets are uniformly randomly distributed after each volley of fire, (b) each unit knows the general area in which enemy units are located but not the consequences of its own fire, and (c) fire from the surviving forces is distributed uniformly over the area in which the enemy forces are located. In the literature equations 5 and 6 are known as Lanchester's linear law formulation.

The state solution is obtained by dividing (5) by (6), integrating, and employing the initial conditions that at $t = 0$, $n = N$ and $m = M$

$$\alpha_A(M - m) = \beta_A(N - n) , \tag{15}$$

which is invariant throughout the battle. If the Red force is annihilated, the associated number of Blue survivors is

$$m = \alpha_A M - \beta_A N \ , \tag{16}$$

which is positive if

$$\alpha_A M > \beta_A N \ . \tag{17}$$

Thus the condition

$$\alpha_A M = \beta_A N \tag{18}$$

implies that both forces will approach zero simultaneously if the battle is described by equations 5 and 6. This formulation suggests that a force's fighting strength is proportional to the force size, giving rise to the name "Lanchester's linear law."

The solution for the number of surviving forces as a function of time is obtained by solving (15) for $\alpha_A m$ and substituting this quantity into (5) producing

$$\frac{dn}{dt} = -(Kn + \beta_A n^2) \ , \tag{19}$$

where $K = (\alpha_A M - \beta_A N)$. Integrating (19),

$$-\log\left|1 + \frac{K}{\beta_A n}\right| = K(-t + C) \ , \tag{20}$$

where C is an arbitrary constant evaluted by the initial condition that n = N at t = 0.

If $(1 + \frac{K}{\beta_A n}) > 0$, from (20)

$$C = -\frac{1}{K}\log[1 + \frac{K}{\beta_A N}] \tag{21}$$

If $(1 + \frac{K}{\beta_A n}) < 0$,

$$C = -\frac{1}{K}\log[-(1 + \frac{K}{\beta_A N})] \; . \tag{22}$$

Substituting either (21) or (22) into (20),

$$\log\left[\frac{1 + \frac{K}{\beta_A N}}{1 + \frac{K}{\beta_A n}}\right] = -Kt$$

and

$$n = \frac{N(\phi - 1)e^{-\beta_A N(\phi-1)t}}{\phi - e^{-\beta_A N(\phi-1)t}} \quad , \tag{23}$$

where $\phi = \alpha_A M/\beta_A N$. Substituting (23) into the state solution (15) and solving,

$$m = \frac{M(\phi - 1)}{\phi - e^{-\beta_A N(\phi-1)t}} \quad . \tag{24}$$

The parameter ϕ is associated with the state solution and expresses the initial advantage of the Blue force over the Red force. This is shown by the ratio

$$\frac{n}{m} = \frac{N}{M} e^{-\beta_A N(\phi-1)t} \quad , \tag{25}$$

which indicates that the Blue force will annihilate the Red force when $\phi > 1$ and will be annihilated when $\phi < 1$.

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Chapter 2

VARIABLE ATTRITION RATES, CONSTANT RATIO MODEL

Seth Bonder and Robert Farrell

In the previous chapter we considered the most straightforward simplification of the basic combat structure--homogeneous forces with constant attrition rates, i.e., attrition rates that are not dependent on battle time or range between the firing weapon and target. Thus the attrition-rate functions (see [B, 1.2 and 1.3]) are constant throughout the battle.

Except for the unlikely situation when neither combatant moves during the course of the battle, the assumption of constant attrition rates is highly unrealistic. Consideration of the acquisition, accuracy, timing, and lethality characteristics explicitly included in the attrition-rate prediction models [B] strongly suggests that the attrition rates would vary with changes in force separation. In this and Chapter 3 we shall consider the effect of this variation in the homogeneous-force battle model with the restriction that the ratio of the attrition-rate functions, $\alpha(r)/\beta(r)$, is constant.[1] This restriction is imposed for analytical purposes in that it facilitates workable closed form solutions that provide some insights into the effect of maneuver in a battle.

[1]The results of the previous chapter will, of course, be a special case of those developed in this one, since the ratio of constant attrition rates is also constant.

2.1 *Battlefield Coordinate System*

As previously noted, the attrition rates will vary when either or both of combatants use mobile weapon systems. The movement of units can be implicitly considered by retaining the battle time dependency in the combat equations or explicitly by converting to a range dependency. Knowledge of the movement schedule provides a one-to-one correspondence between time and range (force separation) during the battle so that they can be, and are, used interchangeably. Use of the range dimension requires the establishment of a coordinate system for the battlefield.

Consider the simplified one-dimensional coordinate system depicted in Figure 1.

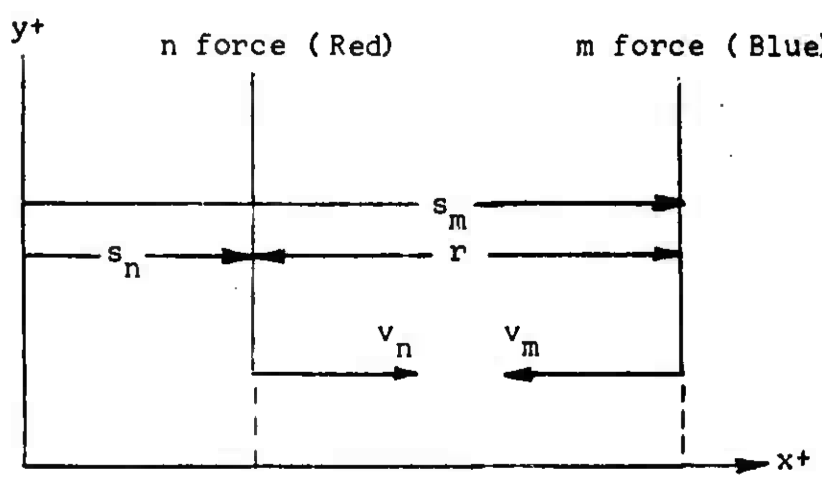

Figure 1. Plan View of Terrain

The distances s_n and s_m are the ranges of the Red and Blue lines, respectively, from a common reference axis. The range between forces at any point in time is denoted by the symbol r. The respective velocities of the Red and Blue force are v_n and

v_m. From the geometry of the figure

$$r = s_m - s_n \qquad s_m > s_n \tag{1}$$

and

$$\frac{dr}{dt} = \frac{ds_m}{dt} - \frac{ds_n}{dt}$$

or

$$v = v_m - v_n \quad , \tag{2}$$

where v is the relative velocity between the Red and Blue forces. An examination of (2) and Figure 1 will indicate that the differential dr has the same sign as v and, accordingly, the type of engagement to be analyzed depends on the values of v_m and v_n.[1] In a meeting engagement $v_n > 0$ and $v_m < 0$ with a resulting rapid decrease in force separation. For an attack engagement $v_n = 0$ and $v_m < 0$. The conditions for a retrograde operation are $v_n > 0$ and $v_m > 0$. If $v_n > v_m$, the range between forces will decrease as the Blue force withdraws. If $v_n < v_m$, the force separation will continuously increase. When $v_n = v_m$ in the retrograde operation, we have the situation described by Weiss (1957) in which the battlefield shifts but the force separation remains constant, i.e., $dr/dt = v = 0$.

The attrition of forces in this homogeneous-force battle model is described by the same equations used in the previous chapter except for the explicit dependency of the attrition

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[1]Engagements are described with the Blue force as reference. That is, an attack engagement considers the Blue force advancing and the Red force defending.

rates on range. Thus

$$\frac{dn}{dt} = -\alpha(r)m \tag{3}$$

and

$$\frac{dm}{dt} = -\beta(r)n\ , \tag{4}$$

where the Blue and Red weapon attrition rates, $\alpha(r)$ and $\beta(r)$, respectively, are now denoted as functions of the force separation r, i.e., the attrition-rate functions. For clarity, however, we shall omit the functional notation throughout most of the developments where omission will not be misleading.

Equations 3 and 4 are used directly in the next section to obtain solutions for the case in which $\alpha(r)/\beta(r)$ is a constant. Explicit range dependency and mobility considerations for the general case in which $\alpha(r)/\beta(r)$ is not constant are added to the description of combat by transforming (3) and (4) from the time to the space domain. From (3)

$$\frac{d}{dt}\left(\frac{dn}{dt}\right) = -\left[\alpha\,\frac{dm}{dt} + m\,\frac{d\alpha}{dt}\right]$$

$$= -\left[\alpha\,\frac{dm}{dr}\,\frac{dr}{dt} + m\,\frac{d\alpha}{dr}\,\frac{dr}{dt}\right]\ ,$$

therefore,

$$\frac{d^2n}{dt^2} = -\left[\alpha\ v\ \frac{dm}{dr} + v\ m\ \frac{d\alpha}{dr}\right] . \tag{5}$$

We also note that

$$\frac{dn}{dt} = v\ \frac{dn}{dr} \tag{6}$$

and

$$\frac{dm}{dt} = v\ \frac{dm}{dr} . \tag{7}$$

Differentiating (6),

$$\frac{d^2n}{dt^2} = v\ \frac{d}{dr}\left(\frac{dn}{dr}\right)\left(\frac{dr}{dt}\right) + \frac{dn}{dr}\ \frac{dv}{dr}\ \frac{dr}{dt}$$

$$= v^2\ \frac{d^2n}{dr^2} + v\ \frac{dv}{dr}\ \frac{dn}{dr}$$

$$= v^2\ \frac{d^2n}{dr^2} + \omega\ \frac{dn}{dr} , \tag{8}$$

where $\omega = v\ \frac{dv}{dr}$ is the relative acceleration between forces. Equating (5) and (8), employing (3), (4), (6), and (7), and rearranging,

$$\frac{d^2n}{dr^2} + \left[\frac{\omega}{v^2} - \frac{1}{\alpha}\ \frac{d\alpha}{dr}\right]\frac{dn}{dr} - \left(\frac{\alpha\beta}{v^2}\right) n = 0 . \tag{9}$$

Analogously,

$$\frac{d^2m}{dr^2} + \left[\frac{\omega}{v^2} - \frac{1}{\beta}\frac{d\beta}{dr}\right]\frac{dm}{dr} - \left(\frac{\alpha\beta}{v^2}\right) m = 0 \ . \qquad (10)$$

Equations 9 and 10 can be used to describe a wide variety of homogeneous-force combat situations. If $\omega = 0$, the equations describe constant-speed engagements. As noted on page 3, the different possible values for $v = (v_m - v_n)$ facilitates describing attack, defense, meeting, and delay engagements, and retrograde operations. Different weapons are considered in terms of the attrition-rate functions $\alpha(r)$ and $\beta(r)$.

The next section of this chapter presents the general time and range solutions to the structure given by equations 1 and 2 and analyzes the effect of a constant assault speed ($\omega = 0$) using linear attrition-rate functions for the Red and Blue weapon systems.

2.2 Time and Range Solutions[1]

Consider a re-write of equations 1 and 2 in which we denote the attrition-rate functions as functions of battle time

$$\frac{dn}{dt} = - \alpha(t)m \qquad (11)$$

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[1] The general solutions described in this section were first presented to the Operations Analysis Techniques working group at the 23rd Military Operations Research Symposium, West Point, New York, June 1969. Solutions to special cases were reported by Bonder (1965).

and

$$\frac{dm}{dt} = -\beta(t)n \quad . \tag{12}$$

We explicitly note the requirement for constancy of the ratio of attrition rates as

$$c = \frac{\beta(t)}{\alpha(t)} = \frac{\beta(0)}{\alpha(0)} = \frac{\beta_o}{\alpha_o} \quad , \tag{13}$$

where

$\alpha(0),[\beta(0)]$ = The Blue [Red] weapon attrition rate when the battle begins at $t = 0$,

$\alpha_o[\beta_o]$ = The Blue [Red] weapon attrition rate when the force separation $r = 0$.

Letting

$$x = \int_0^t \alpha(\tau)d\tau \tag{14}$$

and substituting dx/dt into (11) and (12)

$$\frac{dn}{dx} = -m \tag{15}$$

$$\frac{dm}{dx} = -cn \quad . \tag{16}$$

These are coupled, constant coefficient, differential equations whose solution, using the boundary conditions $n(0) = N$, $m(0) = M$, and $\left.\frac{dn}{dx}\right|_{x=0} = -M$, is given as

$$n(x) = N \cosh(\sqrt{c}\, x) - \frac{1}{\sqrt{c}} M \sinh(\sqrt{c}\, x) \qquad (17)$$

and

$$m(x) = M \cosh(\sqrt{c}\, x) - \sqrt{c}\, N \sinh(\sqrt{c}\, x) . \qquad (18)$$

Rewriting,

$$x = \int_0^t \alpha(\tau)d\tau = \left[\frac{1}{t}\int_0^t \alpha(\tau)d\tau\right] t \qquad (19)$$

$$= \overline{\alpha(t)}\, t ,$$

where $\overline{\alpha(t)}$ is the time average of the attrition-rate function. Substituting for x in (17) and (18),

$$n(t) = N \cosh[\sqrt{c}\, \overline{\alpha(t)}t] - \frac{1}{\sqrt{c}} M \sinh[\sqrt{c}\, \overline{\alpha(t)}t] \qquad (20)$$

and

$$m(t) = M \cosh[\sqrt{c}\, \overline{\alpha(t)}t] - \sqrt{c}\, N \sinh[\sqrt{c}\, \overline{\alpha(t)}t] . \qquad (21)$$

If we consider a constant-speed Blue attack engagement [illegible] defense, i.e., $v_r = 0$, then $v = v_m$ is [illegible]

$$\tau = - \frac{R_o - r}{v} , \tag{22}$$

where

R_o = range at which the battle is initiated.

Thus, the range average of the attrition-rate function from the beginning of the battle to range r can be written as

$$\overline{\alpha(r)} = - \left(\frac{v}{R_o - r}\right) \int_{R_o}^{r} \alpha(s) \frac{ds}{v}$$

$$= - \frac{1}{(R_o - r)} \int_{R_o}^{r} \alpha(s) \, ds \quad . \tag{23}$$

Note that $\alpha(r)$ is also positive for $r > R_o$ and is assumed independent of the assault speed.

With this transformation, the surviving forces as a function of range to the defended position is given directly as

$$n(r) = N \cosh [\theta(r)] + \frac{1}{\sqrt{c}} M \sinh [\theta(r)] \tag{24}$$

and

$$m(r) = M \cosh [\theta(r)] + \sqrt{c} \, N \sinh [\theta(r)] , \tag{25}$$

where

$$\theta(r) = \sqrt{c}\,\overline{\alpha(r)}\left(\frac{R_o - r}{v}\right) \tag{26}$$

is always a negative quantity since $v < 0$ and the other terms are positive.

The state solution, in either the time or space domain, is derived in the same manner as the constant attrition-rate case (see page 168) and given by

$$\alpha_o[M^2 - m^2] = \beta_o[N^2 - n^2]. \tag{27}$$

This is analogous to the classical Lanchester square law, which implies that Blue would lose, i.e., be annihilated, if

$$\alpha_o M^2 < \beta_o N^2. \tag{28}$$

The fallacy of this statement becomes apparent now that we are considering explicit movement of one of the forces. Recognition of the capability to move suggests we consider an end of battle condition which is different from complete annihilation of one force or a draw in which both forces tend to zero simultaneously. A force can counter the lose or draw condition by using its mobility. This is seen in the following discussion which considers specific attrition rate functions for the Blue and Red weapons.

Assume the Blue and Red forces are equipped with weapon systems such that

$$\alpha(r) = \begin{cases} \frac{\alpha_o}{R_e}(R_e - r) & 0 \leq r \leq R_e \\ 0 & r > R_e \end{cases}$$

$$= \begin{cases} K_\alpha(R_e - r) & 0 \leq r \leq R_e \\ 0 & r > R_e \end{cases} \qquad (29)$$

and

$$\beta(r) = \begin{cases} \frac{\beta_o}{R_e}(R_e - r) & 0 \leq r \leq R_e \\ 0 & r > R_e \end{cases}$$

$$= \begin{cases} K_\beta(R_e - r) & 0 \leq r \leq R_e \\ 0 & r > R_e \end{cases} , \qquad (30)$$

where

R_e = the range at which a weapon system first obtairs a nonzero attrition rate

$K_\alpha[K_\beta]$ = slope of the Blue [Red] weapon attrition-rate function .

These attrition-rate functions are shown in Figure 2 along with the starting range parameter for the battle.

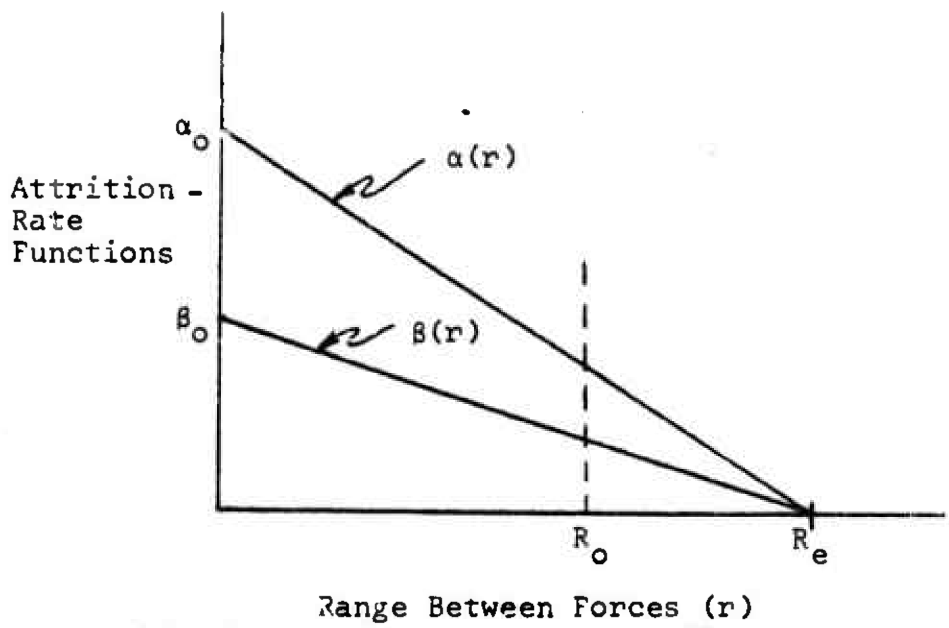

Figure 2 Constant-Ratio, Linear Attrition-Rate Functions

Using the Blue attrition-rate function for values of

$r \leq R_o \leq R_e$,

$$\overline{\alpha(r)} = -\frac{1}{(R_o - r)} \int_{R_o}^{r} K_\alpha (R_e - s)\, ds$$

$$= -\frac{K_\alpha}{2(R_o - r)} \left\{ \left[2R_e s - s^2 \right]_{R_o}^{r} \right\}$$

$$= \frac{K_\alpha}{2(R_o - r)} \left[(R_e - r)^2 - (R_e - R_o)^2 \right]$$

$$= \frac{\alpha_o}{2R_e(R_o - r)} \left[(R_e - r)^2 - (R_e - R_o)^2 \right] . \quad (31)$$

Substituting (31) into (26) gives

$$\theta_\ell(r) = \frac{\sqrt{\alpha_o \beta_o}}{2R_e v} \left[(R_e - r)^2 - (R_e - R_o)^2 \right] \quad (32)$$

and

$$n(r) = N \cosh \theta_\ell + \frac{1}{\sqrt{c}} M \sinh \theta_\ell \quad (33)$$

$$n(r) = M \cosh \theta_\ell + \sqrt{c}\, N \sinh \theta_\ell . \quad (34)$$

The subscript ℓ on θ indicates it is the argument for solutions (24) and (25) when linear attrition-rate functions are appropriate.

With these solutions we can see the impact of mobility by considering the range intervals such that units of the Red and Blue forces survive. Setting $n > 0$ and solving for r in (33), the range interval for the Red force is

$$r > R_e - \sqrt{\frac{-2R_e v}{\sqrt{\alpha_o \beta_o}} \tanh^{-1} \left[\frac{N \sqrt{\beta_o}}{M \sqrt{\alpha_o}} \right] + (R_e - R_o)^2} , \quad (35)$$

and from (24) the range interval for the Blue force is

$$r > R_e - \sqrt{\frac{-2R_e v}{\sqrt{\alpha_o \beta_o}} \tanh^{-1}\left[\frac{M\sqrt{\alpha_o}}{N\sqrt{\beta_o}}\right] + (R_e - R_o)^2} \,. \qquad (36)$$

We define R_j^i as the range at which the j^{th} force has i surviving units. Examination of (35) and (36) will reveal that, if $\sqrt{\alpha_o}\, M < \sqrt{\beta_o} N$, then $R_m^o > R_n^o$, and consequently, the Blue force *could* be destroyed before they reached the Red force defensive line. From (36)

$$R_m^o = R_e - \sqrt{\frac{-2R_e v}{\sqrt{\alpha_o \beta_o}} \tanh^{-1}\left[\frac{M\sqrt{\alpha_o}}{N\sqrt{\beta_o}}\right] + (R_e - R_o)^2} \qquad (37)$$

and, if $\sqrt{\beta_o}\, N > \sqrt{\alpha_o}\, M$,

$$\frac{\partial R_m^o}{\partial v} = \frac{R_e \tanh^{-1}\left(\frac{M}{N}\sqrt{\frac{\alpha_o}{\beta_o}}\right)}{\sqrt{\alpha_o \beta_o}\sqrt{\frac{-2R_e v}{\sqrt{\alpha_o \beta_o}} \tanh^{-1}\left[\frac{M\sqrt{\alpha_o}}{N\sqrt{\beta_o}}\right] + (R_e - R_o)^2}} \qquad (38)$$

$$> 0 \,.$$

Since, however, ∂v is a negative number as $|v|$ increases, R_m^o decreases as speed increases. Therefore, if the attack

were conducted with sufficient speed, the Blue force could overrun the defended objective with some surviving units. This concept of using mobility to saturate the defending line is examined at length in Chapter 3.

2.3 Some Historical Perspectives

Recognition of the capability of a force to move, and consideration of end of battle conditions, adds, in a quantitative manner, another dimension to the classical differential theory of combat, vis, a force can attack with sufficient speed to saturate an enemy's retaliatory capability. In accordance with the classical force-concentration principle, the model indicates that attacking with sufficient speed *and* superiority in numbers is an ideal means of rapidly saturating an enemy's firepower. More importantly, the model suggests that, in the absence of force superiority, an attack with adequate speed is a means of conserving one's own force, i.e., get the enemy before he gets you.

Since it is difficult to conduct experiments during military actions, deductions of this nature are hard to verify. In addition, the unavailability of reliable empirical information regarding past battles (Schroeder, 1963; Helmbold, 1964) precludes quantitative comparison of the model in retrospect. The concept of attacking with appreciable speed to saturate an enemy's retaliatory capability does, however, appear to compare favorably with military experience.

In discussing the offensive employment of tanks, General Bruce C. Clarke noted (1962):

> Always use the maximum number of tanks practicable in the assault. Move fast in the assault. Close fast with the enemy. Fire tank machine guns on the move. The tank casualties you will suffer will vary as the amount of time it takes from the line of departure to the objective. In a tank, 'speed is armor.' Thus the tank tracks, if properly used, are both an offensive weapon and a help in its protection.

During World War II, Field Marshall Rommel frequently employed panzer attacks against larger forces. This is noted by Alfred Gause, Rommel's chief of staff in North Africa (1958):

> The general strength ratios and the supply situation compelled *Rommel* [italics mine] almost always to attack numerically superior forces. Thus, in his attack against Bir-Hacheim-Ain el Gazala positions, where he sought to force a decision, he deliberately opened the offensive on 27 May 1942 with an adverse strength ratio of 6:9 in tanks.

The Sinai campaign (O'Ballance, 1959) describes small-unit engagements in which the victorious Israelis conducted successful attacks in the face of strongly entrenched Egyptian positions. This campaign represents the most recent, but by no means historically isolated, demonstration that saturation of an enemy's retaliatory capability by rapid assault is an important factor in successful combat.

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Chapter 3

THE EFFECT OF MANEUVER: CONSTANT-RATIO ATTRITION-RATE FUNCTIONS

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The previous chapter presented a general solution to the homogeneous-force, differential model of combat with constant-ratio attrition-rate functions. Explicit consideration of assault speed and force separation in a Blue force attack engagement indicated that three outcomes are possible in an engagement of this kind:

(1) Annihilation of either the attacker or defender,

(2) A draw in which both sides tend to zero simultaneously,

(3) The attacking force overruns the defended position.

In this chapter we shall examine the conditions under which the third of these outcomes occurs and, in particular, study in detail the effect that assault speed has on the battle results.

It is reasonable to conjecture that, if the defended position is overrun, the ensuing "close-combat" battle (if one occurs at all) will not adequately be described by our basic differential equation structure. Accordingly, it is of interest to examine the impact that assault speed has on indicators or measures of future success in taking the defended position, where success implies winning the "close-combat" battle at the defended position or having the

defenders retreat before the objective is reached. The measures considered in this analysis are the difference $(m - n)$ and ratio (m/n) of survivors when the attacking force reaches the defended position $(r = 0)$. The effect of assault speed on other measures of success, such as the ratio $(m - n)/(m + n)$ at $r = 0$ or the ratio (m/n) at range r, can be obtained by a directly analogous approach.

Before proceeding it is important to remember that the analysis is based on having a constant ratio of attrition-rate functions. Accordingly, the results should not be interpreted in any absolute sense, but rather to provide some basic insight into the dynamics of combat.

3.1 *Preliminary Results and Notations*

In the preceding chapter we showed that the surviving numbers of units as a function of force separation was given by

$$n(r) = N \cosh [\theta(r)] + \frac{1}{\sqrt{c}} M \sinh [\theta(r)] \qquad (1)$$

and

$$m(r) = M \cosh [\theta(r)] + \sqrt{c}\, N \sinh [\theta(r)] , \qquad (2)$$

where

$$\theta(r) = \sqrt{c}\, \overline{\alpha(r)} \left(\frac{R_o - r}{v}\right) \qquad (3)$$

$$c = \beta(r)/\alpha(r) = \beta_o/\alpha_o$$

$$\overline{\alpha(r)} = -\frac{1}{(R_o - r)} \int_{R_o}^{r} \alpha(s)\, ds \quad . \tag{4}$$

At $r = 0$,

$$n(0) = N \cosh(\theta^o) + \frac{1}{\sqrt{c}} M \sinh(\theta^o) \tag{5}$$

$$m(0) = M \cosh(\theta^o) + \sqrt{c}\, N \sinh(\theta^o) , \tag{6}$$

where

$$\theta^o = \theta(0)$$

$$= \sqrt{c}\, \overline{\alpha(0)} \left(\frac{R_o}{v}\right) \tag{7}$$

$$= C/v \tag{8}$$

since, except for the assault speed, all the terms on the right-hand side of (7) are treated as constants in the analysis. We note that $\theta^o < 0$ and that

$$\frac{\partial \theta^o}{\partial v} = - \frac{\sqrt{c}\, \overline{\alpha(0)} R_o}{v^2}$$

$$= - \frac{\theta^o}{v} < 0 \quad . \tag{9}$$

We also h ve

$$\text{[illegible]} = \left[M \sinh \theta^o + \sqrt{\beta_o/\alpha_o}\, N \cosh \theta^o \right] \frac{\partial \theta^o}{\partial v} \tag{10}$$

$$= \sqrt{\beta_o/\alpha_o} \quad n \frac{\partial\theta^o}{\partial v} \tag{11}$$

by substitution of (1) and

$$\left.\frac{\partial n}{\partial v}\right|_{r=0} = \left[N \sinh \theta^o + \sqrt{\alpha_o/\beta_o} \; M \cosh \theta^o\right] \frac{\partial\theta^o}{\partial v} \tag{12}$$

$$= \sqrt{\alpha_o/\beta_o} \quad m \frac{\partial\theta^o}{\partial v} \; . \tag{13}$$

For the measures $(m - n)$ and (m/n) at $r = 0$ to be meaningful, we must exclude cases in which $m(0) < 0$ and $n(0) < 0$. For $\alpha_o M^2 < \beta_o N^2$, the assault speed which will result in $m(0) = 0$ is obtained by setting (6) equal to zero and solving for

$$v^{m=0} = \frac{-C}{\tanh^{-1}\left[\dfrac{\sqrt{\alpha_o}\; M}{\sqrt{\beta_o}\; N}\right]} \; , \tag{14}$$

where

$$C = \sqrt{c} \; \overline{\alpha(0)} \; R_o \; . \tag{15}$$

Analogously, if $\alpha_o M^2 > \beta_o N^2$, then $n(0) = 0$ at

$$v^{n=0} = \frac{-C}{\tanh^{-1}\left[\dfrac{\sqrt{\beta_o}\; N}{\sqrt{\alpha_o}\; M}\right]} \; . \tag{16}$$

Thus, the defended position will be overrun for assault speeds $-v > -v^{m=0}$ or $-v > -v^{n=0}$, which ever is appropriate. Our concern in this analysis is the effect of assault speed in these intervals.

3.2 Different Attrition-Rate Functions

Analysis in this chapter of the effect of assault speed on the measures $(m - n)$ and (m/n) at $r = 0$ is general in that it can be applied if the ratio of the attrition-rate functions is constant, independent of the shape of the individual attrition rate functions. However, the magnitude of the speed effects will vary when different attrition-rate functions are used. In this section we list a number of attrition-rate functions that have been specifically considered. The functional shapes were suggested by examining the range variation in predicted attrition rates for weapons with widely different characteristics.[1] The constant Lanchester attrition rate is also included.

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[1]This examination was made using arithmetic mean rates, $E(\frac{1}{T})$, before it was shown that the appropriate mean rate to use is the harmonic mean, $1/E(T)$, as proven in [B, 1.2]. Since

$$E(\frac{1}{T}) \geq \frac{1}{E(T)},$$

the reader is cautioned that other functional forms may be more appropriate.

Linear:

$$\alpha_\ell(r) = \begin{cases} \dfrac{\alpha_o}{R_e}(R_e - r) & r \leq R_e \\ 0 & r > R_e \end{cases} \tag{17}$$

$$\overline{\alpha_\ell(0)} = \frac{1}{R_o} \int_0^{R_o} \frac{\alpha_o}{R_e} (R_e - s)\, ds$$

$$= \frac{\alpha_o}{2R_e R_o} (2R_e R_o - R_o{}^2) \tag{18}$$

$$\theta_\ell^o = \frac{\sqrt{\alpha_o \beta_o}}{2R_e\, v} (2R_e R_o - R_o{}^2)$$

$$= \frac{C_\ell}{v} \tag{19}$$

Quadratic:

$$\alpha_q(r) = \begin{cases} \alpha_o(1 - \dfrac{r}{R_e})^2 & r \leq R_e \\ 0 & r > R_e \end{cases} \tag{20}$$

$$\overline{\alpha_q(0)} = \frac{1}{R_o}\int_0^{R_o} \alpha_q(s)\, ds$$

$$= \frac{1}{R_o R_e^{2}} (3R_e^{2}R_o - 3R_eR_o + R_o^{3}) \quad (21)$$

$$\theta_q^o = \frac{\sqrt{\alpha_o\beta_o}}{R_e^{2}\, v} (3R_e^{2}R_o - 3R_eR_o + R_o^{3})$$

$$= \frac{C_q}{v} \quad (22)$$

Cosine:

$$\alpha_c(r) = \begin{cases} \frac{\alpha_o}{2}\left[1 + \cos\left(\frac{\pi r}{R_e}\right)\right] & r \leq R_e \\ 0 & r > R_e \end{cases} \quad (23)$$

$$\overline{\alpha_c(0)} = \frac{1}{R_o}\int_0^{R_o} \alpha_c(s)\, ds$$

$$= \frac{\alpha_o}{2R_c}\left[R_o + \frac{R_e}{\pi}\sin\left(\frac{\pi R_o}{R_e}\right)\right] \quad (24)$$

$$\theta_c^o = \frac{\sqrt{\alpha_o\beta_o}}{2v}\left[R_o + \frac{R_e}{\pi}\sin\left(\frac{\pi R_o}{R_e}\right)\right]$$

$$= \frac{C_c}{v} \tag{25}$$

Exponential:

$$\alpha_e(r) = \begin{cases} \alpha_o(1 - e^{-(R_e - r)}) & r \leq R_e \\ 0 & r > R_e \end{cases} \tag{26}$$

$$\overline{\alpha_e(0)} = \frac{1}{R_o} \int_0^{R_o} \alpha_e(s)\, ds$$

$$= \alpha_o \left[1 - \frac{e^{-R_e}}{R_o} \left(e^{R_o} - 1 \right) \right] \tag{27}$$

$$\theta_e^o = \frac{\sqrt{\alpha_o \beta_o}}{v} R_o \left[1 - \frac{e^{-R_e}}{R_o} \left(e^{R_o} - 1 \right) \right]$$

$$= \frac{C_e}{v} \tag{28}$$

Lanchester:

$$\alpha_L(r) = \begin{cases} \alpha_o & r \leq R_e \\ 0 & r > R_e \end{cases} \tag{29}$$

$$\alpha_L(0) = \frac{1}{R_o}\int_0^{R_o} \alpha_L(s)\, ds$$

$$= \alpha_o \qquad (30)$$

$$\theta_L^o = \frac{\sqrt{\alpha_o \beta_o} R_o}{v}$$

$$= \frac{C_L}{v} \qquad (31)$$

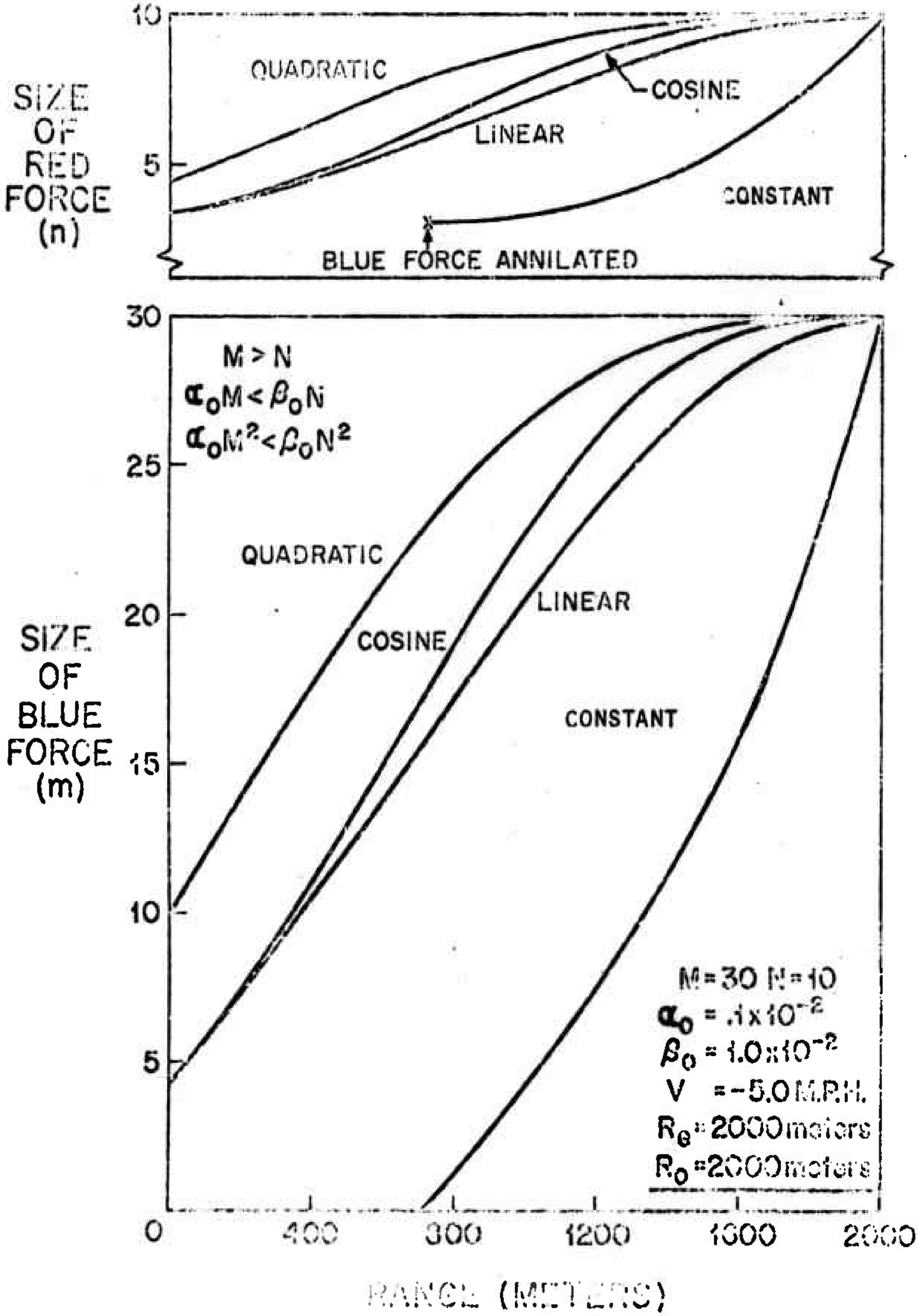
The surviving numbers of forces for some of the different attrition-rate functions are compared in Figure 1. These are obtained by direct substitution of the appropriate attrition-rate function in (4), (4) into (3), and then (3) into (2) and (1). The marked differences between the formulations are evident--especially between the variable attrition-rate formulations and the Lanchester constant attrition-rate one. For example, the constant attrition-rate solution predicts annihilation of the Blue force at 700 meters with two remaining Red units, while use of quadratic attrition rate function would predict ten Blue and four Red surviving units. As shown in Figure 3 these differences are reduced when the engagement range (R_o) is much less than the effective range of

the weapon (R_e), and, in the limit as the weapon's effective range approaches infinity, the solutions converge to the one with constant attrition-rate functions, $\alpha_L(r)$. This obtains since the differences in the solutions are solely dependent on the form of $\theta(r)$ and

$$\lim_{R_e \to \infty} \theta_\ell(r) = \lim_{R_e \to \infty} \theta_q(r) = \lim_{R_e \to \infty} \theta_c(r) = \lim_{R_e \to \infty} \theta_e(r) = \theta_L(r) \ .$$

It is of interest to point out that the large effect of the assault speed (noted in the last chapter and in following sections of this one) and the difference $R_e - R_c$ (noted above) on the numbers of surviving forces may explain some of the conflicting conclusions of studies to verify the classical Lanchester theory via the correlation between observed and theoretical attrition histories of battles.[1] If a battle were fought without appreciable movement or if R_o were appreciably less than R_e, observed attrition data might correlate with predicted attrition of forces in a battle regardless of the particular weapon characteristics, i.e., attrition-rate functions. If, however, the forces employed moving weapons, and $R_o \approx R_e$, failure to explicitly consider specific variations in weapon attrition rates with range might readily produce large deviations between observed and predicted force attrition.

[1] For example, Engel (1954), Weiss (1957), and Willard (1962).



[illegible] Surviving Numbers of Forces for Different Attrition Rate Functions

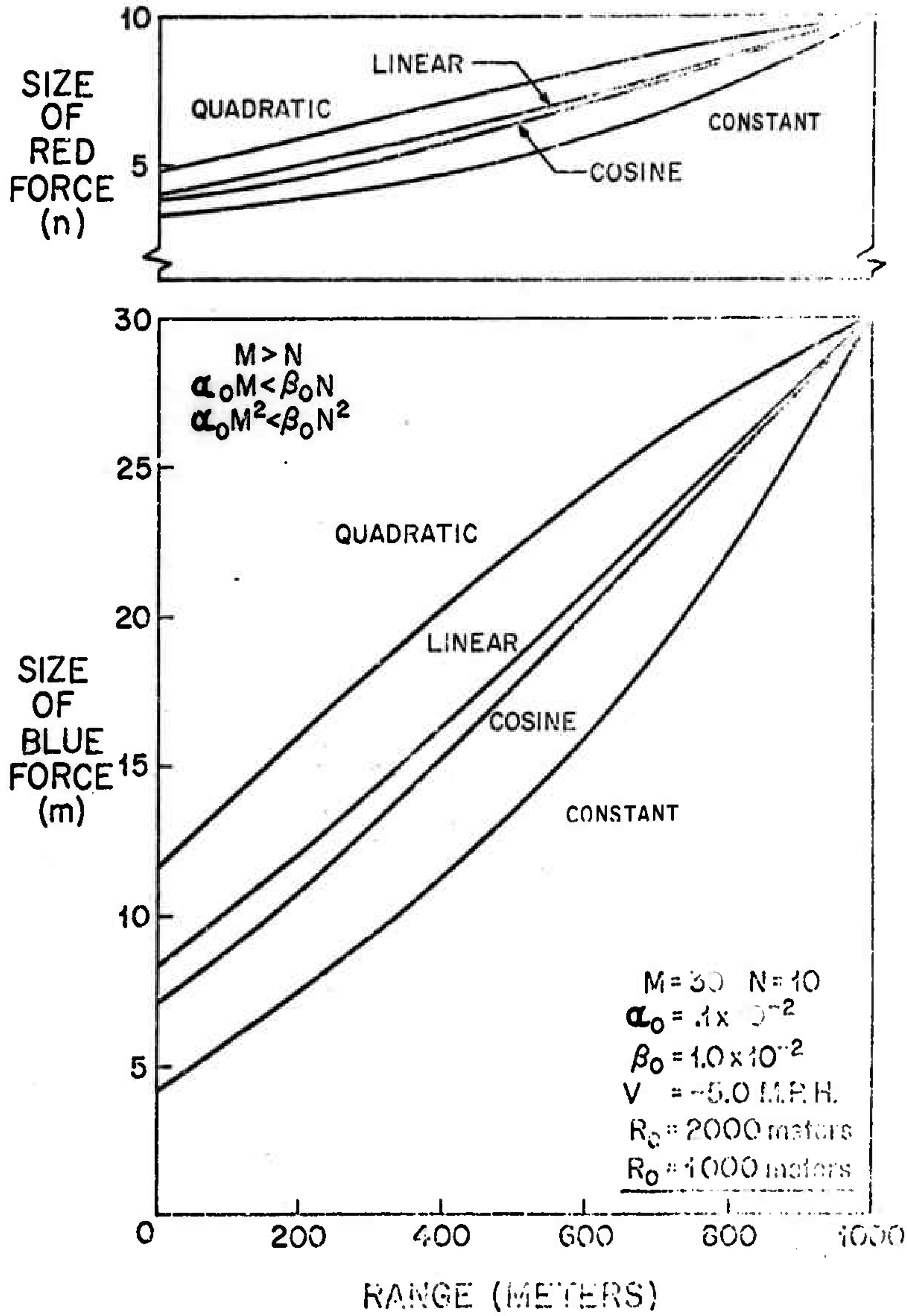


Figure 2 Comparison of Surviving Numbers of Forces for Different Attrition Rate Functions

3.3 The Difference (m - n) at the Defended Position

In this section we consider the difference (m - n) at $r = 0$, which we denote as d_o. For assault speeds $-v < -v^{m=0}$ or $-v < -v^{n=0}$, as appropriate, d_o is a constant . This is seen by considering $\alpha_o M^2 < \beta_o N^2$. Then for $-v \leq -v^{m=0}$, $m = 0$ for some $r \geq 0$ and

$$n_{m=0} = N \cosh \phi + \sqrt{\alpha_o / \beta_o}\, M \sinh \phi \,, \qquad (32)$$

where

$$\phi = -\tanh^{-1} \left[\frac{M \sqrt{\alpha_o}}{N \sqrt{\beta_o}} \right] \qquad (33)$$

is obtained by setting (2) equal to zero. This implies that $\theta^o = \phi$, which is a constant. Thus, for $-v \leq -v^{m=0}$, $d_o = -n_{m=0}$, which is a constant. Similarly, for $\alpha_o M^2 > \beta_o N^2$ and $-v \leq -v^{n=0}$, $d_o = m_{n=0}$, which is a constant such that

$$m_{n=0} = M \cosh \psi + \sqrt{\beta_o / \alpha_o}\, N \sinh \psi \,, \qquad (34)$$

where

$$\psi = -\tanh^{-1} \left[\frac{N \sqrt{\beta_o}}{M \sqrt{\alpha_o}} \right] \qquad (35)$$

is obtained by setting (1) equal to zero. Thus, for $0 \leq -v \leq -v^{m=0}$ or $-v^{n=0}$, $\partial d_o / \partial v = 0$. The value of d_o for

this speed interval is constant for all attrition-rate functions and depends only on M, N, α_o, β_o,

Substracting (5) from (6)

$$d_o = (M - N) \cosh \theta^o + \frac{(\beta_o N - \alpha_o M)}{\sqrt{\alpha_o \beta_o}} \sinh \theta^o . \qquad (36)$$

Setting $d_o = 0$ implies

$$(M - N) \cosh \theta^o = \frac{(\beta_o N - \alpha_o M)}{\sqrt{\alpha_o \beta_o}} (-\sinh \theta^o) \qquad (37)$$

$$-\tanh \theta^o = \frac{(M - N)\sqrt{\alpha_o \beta_o}}{(\beta_o N - \alpha_o M)}$$

and since $\theta^o < 0$,

$$0 < \frac{(M - N) \sqrt{\alpha_o \beta_o}}{(\beta_o N - \alpha_o M)} < 1 . \qquad (38)$$

Suppose $(M - N) > 0$, then $(\beta_o N - \alpha_o M) > 0$ and

$$(M - N)\sqrt{\alpha_o \beta_o} < \beta_o N - \alpha_o M ,$$

which implies $\alpha_o M^2 < \beta_o N^2$. Similarly, the assumption that $(M - N) < 0$ implies that $\alpha_o M^2 > \beta_o N^2$. Accordingly,

$d_o = 0$ for $(M - N)(\beta_o N - \alpha_o M) > 0$ and

$$\alpha_o M^2 < \beta_o N^2 \quad \text{if } (M - N) > 0$$

$$\alpha_o M^2 > \beta_o N^2 \quad \text{if } (M - N) < 0 \; .$$

Thus, $d_o = 0$ for two sets of initial conditions

Condition I: Condition II:

$$\begin{array}{rcl} (M - N) > 0 & \qquad & (M - N) < 0 \\ \alpha_o M - \beta_o N < 0 & & \alpha_o M - \beta_o M > 0 \\ \alpha_o M^2 - \beta_o N^2 < 0 & & \alpha_o M^2 - \beta_o N^2 > 0 \end{array} \tag{39}$$

The speed that results in $d_o = 0$ is obtained from (36) by

$$- v^{m=n} = \frac{C}{\tanh^{-1}\left[\dfrac{(M - N)\sqrt{\alpha_o \beta_o}}{\beta_o N - \alpha_o M}\right]} \; . \tag{40}$$

Consider the quantity

$$D = \frac{M\sqrt{\alpha_o}}{N\sqrt{\beta_o}} - \frac{(M - N)\sqrt{\alpha_o \beta_o}}{\beta_o N - \alpha_o M}$$

$$= \frac{\sqrt{\alpha_o}(\beta_o N^2 - \alpha_o M^2)}{\sqrt{\beta_o}\, N\, (\beta_o N - \alpha_o M)} \; , \tag{41}$$

So far we have considered the cases in which d_o equals a constant ($-n_{m=0}$ or $m_{n=0}$) and $d_o = 0$. We next examine the sign of d_o when it is not a constant, i.e., $-v > -v^{n=0}$ or $-v^{m=0}$ as appropriate. Examination of (36) leads directly to:

I. If $M = N$,

(i) $\beta_o N - \alpha_o M > 0 \Longrightarrow m - n < 0$

(ii) $\beta_o N - \alpha_o M < 0 \Longrightarrow m - n > 0$.

II. If $M > N$,

(i) $\beta_o N - \alpha_o M < 0 \Longrightarrow m - n > 0$

(ii) $\beta_o N - \alpha_o M = 0 \Longrightarrow m - n > 0$.

III. If $M < N$,

(i) $\beta_o N - \alpha_o M > 0 \Longrightarrow m - n < 0$

(ii) $\beta_o N - \alpha_o M = 0 \Longrightarrow m - n < 0$.

This leaves the following cases for consideration:

IV. $M > N$ and $(\beta_o N - \alpha_o M) > 0$

(i) $\alpha_o M^2 > \beta_o N^2$

(ii) $\alpha_o M^2 < \beta_o N^2$

V. $M < N$ and $(\beta_o N - \alpha_o M) < 0$

(i) $\alpha_o M^2 > \beta_o N^2$

(ii) $\alpha_o M^2 < \beta_o N^2$.

Cases IV(i) and V(ii)

Consider Case IV(i). Suppose $d_o < 0$, then from (36) this implies

$$\frac{(M - N)\sqrt{\alpha_o \beta_o}}{(\beta_o N - \alpha_o M)} < -\tanh \theta^o$$

$$0 < \frac{(M - N)\sqrt{\alpha_o \beta_o}}{(\beta_o N - \alpha_o M)} < 1 \quad ,$$

$$(M - N)\sqrt{\alpha_o \beta_o} < \beta_o N - \alpha_o M \quad ,$$

or

$$\alpha_o M^2 < \beta_o N^2 \quad ,$$

which is counter to the assumption in this case. Thus $d_o \not< 0$ and, by reversing the inequality, it is easily shown that, in

fact, $d_o > 0$ for Case IV i. By a directly analogous argument it can also be shown that $d_o < 0$ for Case V(ii).

Cases IV(ii) and V(i)

Both these cases lead to $d_o = 0$ if $v = v^{m=n}$ given by (40). Consider IV(ii) and suppose $d_o < 0$. From (36) this implies

$$-\theta^o > \tanh^{-1}\left[\frac{(M - N)\sqrt{\alpha_o\beta_o}}{\beta_o N - \alpha_o M}\right]$$

and

$$-v < \frac{C}{\tanh^{-1}\left[\frac{(M - N)\sqrt{\alpha_o\beta_o}}{(\beta_o N - \alpha_o M)}\right]} = -v^{m=n} .$$

For $d_o > 0$ it follows that

$$-v > -v^{m=n} .$$

In an analogous fashion for Case V(i), it can be shown that

$$d_o < 0 \text{ if } -v > -v^{m=n}$$

and

$$d_o > 0 \text{ if } -v < -v^{m=n} .$$

A summary of the effects of variations in M, N, α_o, β_o on d_o are shown in Table 1. The value of d_o depends on the signs of

Initial Conditions: $M - N$

Linear Conditions: $\alpha_o M - \beta_o N$

Quadratic Conditions: $\alpha_o M^2 - \beta_o N^2$.

The condition $\alpha_o M^2 = \beta_o N^2$ is also included in Table 1. Substituting for M and N in (5) and (6), respectively,

$$n(0) = N\, e^{\theta^o}$$

and

$$m(0) = M\, e^{\theta^o} .$$

For $M \neq N$,

$$d_o = (M - N)\, e^{\theta^o} .$$

Thus, d_o has the same sign as $(M - N)$ and cannot be zero for finite v.

Finally, we note that as $v \to -\infty$, $d_o \to M - N$. This intuitively obvious result obtains from (36), where

$$\lim_{v\to-\infty} d_o = (M - N) \lim_{v\to-\infty} \cosh \theta^o + \frac{\beta_o N - \alpha_o M}{\sqrt{\alpha_o \beta_o}} \lim_{v\to-\infty} \sinh \theta^o$$

$$= M - N$$

since $\theta^o = C/v$.

Table 1 The Difference d_o As a Function of M, N, α_o, β_o

$M - N$	$\alpha_o M - \beta_o N$	$\alpha_o M^2 - \beta_o N^2$	$d^o = (m - n)$ at $r = 0$
0	<0	<0	<0
0	>0	>0	>0
>0	>0	>0	>0
>0	0	>0	>0
>0	<0	>0	>0
>0	<0	<0	$\begin{Bmatrix} <0 \\ 0 \\ >0 \end{Bmatrix} - v \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \dfrac{C}{\tanh^{-1}\left[\dfrac{(M-N)\sqrt{\alpha_o\beta_o}}{\beta_o N - \alpha_o M}\right]}$
<0	<0	<0	<0
<0	0	<0	<0
<0	>0	<0	<0
<0	>0	>0	$\begin{Bmatrix} <0 \\ 0 \\ >0 \end{Bmatrix} - v \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \dfrac{C}{\tanh^{-1}\left[\dfrac{(M-N)\sqrt{\alpha_o\beta_o}}{\beta_o N - \alpha_o M}\right]}$
0	0	0	0
>0	<0	0	>0
<0	>0	0	<0

which is positive for condition I in (39). Hence,

$$\tanh^{-1}\left[\frac{M\sqrt{\alpha_o}}{N\sqrt{\beta_o}}\right] > \tanh^{-1}\left[\frac{(M - N)\sqrt{\alpha_o\beta_o}}{\beta_o N - \alpha_o M}\right] , \qquad (42)$$

which, by comparing (14) to (40), implies

$$-v^{m=0} < -v^{m=n} . \qquad (43)$$

Consideration of condition II in (39) analogously implies

$$-v^{n=0} < -v^{m=n} . \qquad (44)$$

We note that for $d_o = 0$, $m = n$, θ^o is fixed, and from (5) and (6)

$$\begin{aligned} m = n &= M \cosh(\Omega) + \sqrt{\beta_o/\alpha_o}\, N \sinh(\Omega) \\ &= N \cosh(\Omega) + \sqrt{\alpha_o/\beta_o}\, M \sinh(\Omega) , \end{aligned} \qquad (45)$$

where

$$\Omega = -\tanh^{-1}\left[\frac{(M - N)\sqrt{\alpha_o\beta_o}}{\beta_o N - \alpha_o M}\right] . \qquad (46)$$

Thus, the constant $m = n$ is independent of the form of the attrition-rate function.

3.4 The Derivatives of d_o

In this section we examine the behavior of the derivatives of d_o with respect to the assault speed v. From (36)

$$d_o' = \frac{\partial d_o}{\partial v}$$

$$= \left\{(M - N)\sinh\theta^o + \frac{(\beta_o N - \alpha_o M)}{\sqrt{\alpha_o\beta_o}}\cosh\theta^o\right\}\frac{\partial\theta^o}{\partial v} \,. \quad (47)$$

Consider first the cases in which $d_o' = 0$. Setting (47) equal to zero,

$$-\tanh\theta^o = \frac{\beta_o N - \alpha_o M}{(M - N)\sqrt{\alpha_o\beta_o}}\,, \qquad (48)$$

which implies

$$0 < \frac{\beta_o N - \alpha_o M}{(M - N)\sqrt{\alpha_o\beta_o}} < 1 \,.$$

If

$$\begin{aligned}(M - N) &> 0\\ \beta_o N - \alpha_o M &> 0\,,\end{aligned} \qquad (49)$$

then from (49) this implies $\alpha_o M^2 > \beta_o N^2$. If

$$(M - N) < 0$$
$$\beta_o N - \alpha_o M < 0, \tag{50}$$

then this implies $\alpha_o M^2 < \beta_o N^2$.

Employing (10) and (11), d_o' can be written as

$$d_o' = \left\{\sqrt{\beta_o/\alpha_o}\; n - \sqrt{\alpha_o/\beta_o}\; m\right\} \frac{\partial \theta^o}{\partial v} \tag{51}$$

and

$$d_o'' = \frac{\partial^2 d_o}{\partial v^2}$$

$$= (m - n)\left(\frac{\partial \theta^o}{\partial v}\right)^2 + \left[\sqrt{\beta_o/\alpha_o}\; n - \sqrt{\alpha_o/\beta_o}\; m\right] \frac{\partial^2 \theta^o}{\partial v^2}$$

$$= (m - n)\left(\frac{\partial \theta^o}{\partial v}\right)^2 - \frac{2}{v}\, d_o' \tag{52}$$

since

$$\frac{\partial^2 \theta^o}{\partial v^2} = \frac{2C}{v^3} = -\frac{2}{v}\frac{\partial \theta}{\partial v} . \tag{53}$$

For the conditions specified by (49), $\alpha_o M^2 > \beta_o N^2$, $d_o' = 0$ and from Table [illegible], $d_o > 0$. Therefore, $d_o'' > 0$ which indicates that d_o is a minimum at the speed for which $d_o' = 0$. In a

directly analogous fashion, the conditions of (50) (and the implied $\alpha_o M^2 < \beta_o N^2$) suggest that $d_o'' < 0$. Thus, d_o has a maximum at the speed for which $d_o' = 0$. The speed which results in $d_o' = 0$ is obtained by setting (47) equal to zero and solving for

$$-v^{d_o'=0} = \frac{C}{\tanh^{-1}\left[\frac{\beta_o N - \alpha_o M}{(M - N)\sqrt{\alpha_o \beta_o}}\right]} \qquad (54)$$

using the condition (49) or (50). It is shown in Appendix C, 3 that the limit of d_o' is zero as the assault speed approaches infinity whether or not $d_o' = 0$ for lower assault speeds.

Consider next the case in which $d_o' \neq 0$. Since $\partial\theta^o/\partial v < 0$, we have directly from (47)

$M - N = 0$

(i) $\alpha_o M - \beta_o N < 0 \implies \frac{\partial m-n}{\partial v} < 0$

(ii) $\alpha_o M - \beta_o N > 0 \implies \frac{\partial m-n}{\partial v} > 0$

$M - N > 0$

(i) $\alpha_o M - \beta_o N > 0 \implies \frac{\partial m-n}{\partial v} > 0$

(ii) $\alpha_o M - \beta_o N = 0 \implies \frac{\partial m-n}{\partial v} > 0$

$M - N < 0$

(i) $\alpha_o M - \beta_o N < 0 \implies \frac{\partial m-n}{\partial v} < 0$

(ii) $\alpha_o M - \beta_o N = 0 \implies \frac{\partial m-n}{\partial v} < 0$.

This leaves four cases for consideration. The first is

I. $M - N > 0$

$\alpha_o M - \beta_o N < 0$

$\alpha_o M^2 - \beta_o N^2 > 0$.

This is condition (49), which leads to $d_o' = 0$ for the assault speed given by (54). For $d_o' < 0$, (47) leads to

$$-v > \frac{C}{\tanh^{-1}\left[\frac{\beta_o N - \alpha_o M}{(M - N)\sqrt{\alpha_o \beta_o}}\right]} = -v^{d_o'=0} \qquad (55)$$

and for $-v < -v^{d_o'=0}$, $d_o' > 0$. In a directly analogous fashion the case[1]

II. $M - N < 0$

$\alpha_o M - \beta_o N > 0$

$\alpha_o M^2 - \beta_o N^2 < 0$

leads to

$$-v < -v^{d_o'=0} \implies d_o' < 0$$

$$-v > -v^{d_o'=0} \implies d_o' > 0 .$$

[1] This case is condition (50).

The third case is

III. $M - N > 0$

$$\alpha_o M - \beta_o N < 0$$

$$\alpha_o M^2 - \beta_o N^2 < 0 \ .$$

Consider (47) and suppose

$$(M - N) \sinh \theta^o + \frac{\beta_o N - \alpha_o M}{\sqrt{\alpha_o \beta_o}} \cosh \theta^o < 0 \ . \qquad (56)$$

This implies

$$0 < \frac{\beta_o N - \alpha_o M}{(M - N)\sqrt{\alpha_o \beta_o}} < 1$$

$$\Longrightarrow \ \beta_o N^2 < \alpha_o M^2 \ ,$$

which is contrary to the above assumption. Since the left-hand side of (56) is not equal to zero under the state conditions, it must be greater than zero, which implies that $d_o' < 0$. The fourth case

IV. $M - N < 0$

$$\alpha_o M - \beta_o N > 0$$

$$\alpha_o M^2 - \beta_o N^2 > 0$$

is analyzed in a directly analogous way to Case III and implies $d_o' > 0$.

The two conditions specified by (49) and (50) have assault speeds such that $d_o' = 0$. For (49) we have $v^{n=0}$ given by (16). The difference

$$D_1 = \frac{N}{M}\sqrt{\beta_o/\alpha_o} - \frac{\beta_o N - \alpha_o M}{\sqrt{\alpha_o\beta_o}\;(M - N)} \tag{57}$$

$$= \frac{\sqrt{\alpha_o}\,\beta_o MN - \sqrt{\alpha_o}\,\beta_o N^2 - \sqrt{\alpha_o}\,\beta_o MN + \sqrt{\alpha_o{}^3}\,M^2}{\sqrt{\alpha_o}\,M(M - N)\sqrt{\alpha_o\beta_o}}$$

$$= \frac{\sqrt{\alpha_o}\,(\alpha_o M^2 - \beta_o N^2)}{\sqrt{\alpha_o}\,M(M - N)\sqrt{\alpha_o\beta_o}} > 0 \; .$$

Therefore,

$$\frac{N}{M}\sqrt{\beta_o/\alpha_o} > \frac{\beta_o N - \alpha_o M}{\sqrt{\alpha_o\beta_o}\;(M - N)}$$

$$\tanh^{-1}\left[\frac{N}{M}\sqrt{\beta_o/\alpha_o}\right] > \tanh^{-1}\left[\frac{\beta_o N - \alpha_o M}{(M - N)\sqrt{\alpha_o\beta_o}}\right]$$

and

$v^{n=0}$ [illegible] $v^{d_o'=0}$.

In a similar fashion from (50), $-v^{m=0} < -v^{d'_o=0}$, where $-v^{m=0}$ is given by (15).

Finally, we note that at $d'_o = 0$

$$m = M \cosh \chi + \sqrt{\beta_o/\alpha_o}\, N \sinh \chi \tag{58}$$

$$n = N \cosh \chi + \sqrt{\alpha_o/\beta_o}\, M \sinh \chi , \tag{59}$$

where

$$\chi = - \tanh^{-1} \left[\frac{\beta_o N - \alpha_o M}{(M - N)\sqrt{\alpha_o \beta_o}} \right] . \tag{60}$$

The constants in (60) are independent of the form of the attrition-rate functions employed.

Table 2 summarizes the results of this and the previous section. The different cases have been numbered to correspond to the numerical examples given in Section 3.6.

Table 2 Final Force Difference for Values of the Initial, Linear, and Quadratic Conditions

Case	M-N	$\alpha_o M-\beta_o N$	$\alpha_o M^2-\beta_o N^2$	$d_o = (m-n)$ at $r = 0$	$d'_o = \frac{\partial(m-n)}{\partial v}\Big\rvert_{r=0}$
1	0	<0	<0	<0	<0
2	0	>0	>0	>0	>0
3	>0	>0	>0	>0	>0
4	>0	0	>0	>0	>0
5	>0	<0	<0	$\begin{Bmatrix}<0\\0\\>0\end{Bmatrix} -v \begin{Bmatrix}<\\=\\>\end{Bmatrix} \dfrac{\sqrt{\alpha_o\beta_o}(2R_eR_o - R_o^2)}{2\tanh^{-1}\left[\dfrac{(M-N)\sqrt{\alpha_o\beta_o}}{\beta_o N - \alpha_o M}\right]}$	<0
6	>0	<0	>0	>0	$\begin{Bmatrix}<0\\0\\>0\end{Bmatrix} -v \begin{Bmatrix}>\\=\\<\end{Bmatrix} \dfrac{\sqrt{\alpha_o\beta_o}(2R_eR_o - R_o^2)}{2\tanh^{-1}\left[\dfrac{\beta_o N - \alpha_o M}{(M-N)\sqrt{\alpha_o\beta_o}}\right]}$
7	<0	<0	<0	<0	<0
8	<0	0	<0	<0	<0
9	<0	>0	>0	$\begin{Bmatrix}<0\\0\\>0\end{Bmatrix} -v \begin{Bmatrix}>\\=\\<\end{Bmatrix} \dfrac{\sqrt{\alpha_o\beta_o}(2R_eR_o - R_o^2)}{2\tanh^{-1}\left[\dfrac{(M-N)\sqrt{\alpha_o\beta_o}}{\beta_o N - \alpha_o M}\right]}$	>0
10	<0	>0	<0	<0	$\begin{Bmatrix}<0\\=0\\>0\end{Bmatrix} -v \begin{Bmatrix}<\\=\\>\end{Bmatrix} \dfrac{\sqrt{\alpha_o\beta_o}(2R_eR_o - R_o^2)}{2\tanh^{-1}\left[\dfrac{\beta_o N - \alpha_o M}{(M-N)\sqrt{\alpha_o\beta_o}}\right]}$
11	0	0	0	0	0
12	<0	<0	0	>0	<0
13	>0	>0	0	<0	>0

3.5 *The Ratio m/n at the Defended Position*

In this section we consider the ratio m/n at $r = 0$, which we denote as ρ_o. The derivative

$$\rho_o' = \frac{\partial \rho_o}{\partial v} = \frac{n\frac{\partial m}{\partial v} - m\frac{\partial n}{\partial v}}{n^2}$$

$$= \frac{1}{n^2}\left\{\sqrt{\beta_o/\alpha_o}\; n^2 - \sqrt{\alpha_o/\beta_o}\; m^2\right\}\frac{\partial \theta^o}{\partial v}$$

when (10) and (11) are employed. Since $\partial\theta^o/\partial v < 0$, the condition that $\rho_o' > 0$ implies $\beta_o n^2 - \alpha_o m^2 < 0$ at $r = 0$. Substituting (5) and (6) intc this condition

$$\beta_o\left[N^2 \cosh^2\theta + 2\sqrt{\alpha_o/\beta_o}\; MN \sinh\theta \cosh\theta + \frac{\alpha_o}{\beta_o} M^2 \sinh^2\theta\right]$$

$$< \alpha_o\left[M^2 \cosh^2\theta + 2\sqrt{\beta_o/\alpha_o}\; MN \sinh\theta\cosh\theta + \frac{\beta_o}{\alpha_o} N^2 \sinh^2\theta\right]$$

$$\beta_o N^2 [\cosh^2\theta - \sinh^2\theta] < \alpha_o M^2 [\cosh^2\theta - \sinh^2\theta]$$

$$\Longrightarrow \beta_o N^2 < \alpha_o M^2 .$$

In a similar manner

$$\rho_o' = 0 \Longrightarrow \beta_o N^2 = \alpha_o M^2$$

and

$$\rho_o' < 0 \Longrightarrow \beta_o N^2 > \alpha_o M^2 .$$

These derivatives plus the fact that a $d_o > 0$ implies $\rho_o > 1$ lead directly to the results found in Table 3.

3.6 Some Numerical Examples--Linear Attrition-Rate Functions

This section presents a verbal description and some specific numerical examples to amplify the mathematical results developed in Sections 3.3, 3.4, and 3.5. The examples employ linear attrition-rate functions from the Blue and Red weapons.[1] The conditions considered correspond to the cases listed in Table 2 and are principally concerned with situations in which the Blue attack force overruns the defended position, i.e., the assault speed is greater than some critical speed.

In overrunning the objective the attacker would obviously desire to do so with maximum $d_o = (m - n)$ at $r = 0$. Since $v < 0$ and $|v|$ is increasing, $\partial v < 0$, and it is advantageous for the attacking Blue force to have $d_o' < 0$. This implies that $\partial(m - n) > 0$ or that the increase in Blue survivors is greater than the increase in Red survivors.

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[1]See Section 3.2 for the specific attrition-rate functions.

Table 3 Final Force Ratio for Values of the Initial, Linear, and Quadratic Superiority Conditions

$M-N$	$\alpha_o M - \beta_o N$	$\alpha_o M^2 - \beta_o N^2$	$\rho_o' = \left.\frac{\partial m/n}{\partial v}\right\|_{r=0}$	$\rho_o = m/n$ at $r = 0$
0	<0	<0	<0	<1
0	>0	>0	>0	>1
>0	>0	>0	>0	>1
>0	>0	>0	>0	>1
>0	<0	>0	>0	>1
>0	<0	<0	<0	$\begin{Bmatrix} <1 \\ 1 \\ >1 \end{Bmatrix} \quad -v \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \dfrac{\sqrt{\alpha_o \beta_o}(2R_e R_o - R_o^2)}{2 \tanh^{-1}\left[\dfrac{(M-N)\sqrt{\alpha_o \beta_o}}{\beta_o N - \alpha_o M}\right]}$
<0	<0	<0	<0	<1
<0	0	<0	<0	<1
<0	>0	<0	<0	<1
<0	>0	>0	>0	$\begin{Bmatrix} <1 \\ 1 \\ >1 \end{Bmatrix} \quad -v \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \dfrac{\sqrt{\alpha_o \beta_o}(2R_e R_o - R_o^2)}{2 \tanh^{-1}\left[\dfrac{(M-N)\sqrt{\alpha_o \beta_o}}{\beta_o N - \alpha_o M}\right]}$
0	0	0	0	1
>0	<0	0	0	>1
<0	>0	0	0	<1

Case 1 $M = N$

$$\alpha_o M - \beta_o N < 0$$

$$\alpha_o M^2 - \beta_o N^2 < 0 \, .$$

Blue is linearly inferior and, coupled with the initial equality, is quadratically inferior. For $-v \leq -v^{m=0}$ the attacking force is annihilated at some $r \geq 0$. For $-v > -v^{m=0}$, Blue overruns the Red defensive line, but is always inferior ($d_o < 0$). Since $d_o' < 0$, Blue's inferiority decreases as the attack speed increases. Minimum d_o occurs for $-v \leq -v^{m=0}$ and increases to zero as speed increases.

Case 2 $M = N$

$$\alpha_o M - \beta_o N > 0$$

$$\alpha_o M^2 - \beta_o N^2 > 0 \, .$$

Blue has linear superiority and, coupled with the initial equality, is quadratically superior. For $-v > -v^{n=0}$ Blue has terminal superiority. Since $d_o' \geq 0$, as the attacking force's speed increases, its superiority at $r = 0$ decreases. Maximum superiority occurs for $-v \leq -v^{n=0}$ and decreases to zero as speed increases.

Cases 3 and 4 $M - N > 0$

$$(3)\ \alpha_o M - \beta_o N > 0$$

$$(4)\ \alpha_o M - \beta_o N = 0\ .$$

In both cases Blue has quadratic superiority. Hence, for $-v \leq -v^{n=0}$, the defending Red force is annihilated by Blue at some $r \geq 0$. For $-v \geq -v^{n=0}$ the Blue force overruns the defensive position, always with terminal superiority, $d_o > 0$. In this case, $d_o' > 0$ and d_o decreases as speed increases. Maximum d_o occurs at $-v \leq -v^{n=0}$ and decreases to $M - N$ as $-v$ increases.

Case 5 $M - N > 0$

$$\alpha_o M - \beta_o N < 0$$

$$\alpha_o M^2 - \beta_o N^2 < 0\ .$$

In this situation the attacking Blue force will be annihilated at $r \geq 0$ if $-v \leq -v^{m=0}$. For $-v > -v^{m=0}$ the Blue force will overrun the Red defensive line. Since $d_o' < 0$, the difference d_o increases as speed increases. Choice of v is much more critical in this situation since

$$d_o \begin{Bmatrix} <0 \\ 0 \\ >0 \end{Bmatrix} \text{ for } -v \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \frac{C}{\tanh^{-1}\left[\dfrac{(M - N)\sqrt{\alpha_o \beta_o}}{\beta_o N - \alpha_o M}\right]}\ .$$

That is, v determines if Blue is terminally superior, equal, or inferior. Minimum d_o (<0) occurs for $-v \leq -v^{m=0}$. The difference d_o increases to $M - N$ as speed increases for $-v > -v^{m=0}$.

Figure 3 is a graph of d_o as a function of assault speed for specific values of M, N, α_o, β_o corresponding to case 5. The linear attrition-rate function has been used with R_e = 2000 meters. The battle starts at R_o = 2000 meters. Note that in this situation, by appropriate choice of speed Blue not only avoids annihilation but also ensures numerical superiority at $r = 0$.

Figures 4 and 5 are graphs of d_o' and ρ_o for the same situation as Figure 3. We note that the "return" from an increase in speed is diminishing. This can also be seen in Figure 3, that is, $d_o' > 0$ and tends monotonically to zero. The graph of ρ_o indicates a rapid increase to about the initial ratio M/N.

Case 6

$$M - N > 0$$

$$\alpha_o M - \beta_o N < 0$$

$$\alpha_o M^2 - \beta_o N^2 > 0\ .$$

In this engagement for $-v \leq -v^{n=0}$ the defending Red force will be annihilated by Blue at some $r \geq 0$. For $-v > -v^{n=0}$ Blue always has terminal superiority, $d_o > 0$. Maximum d_o occurs for $-v < -v^{n=0}$. In this situation, however, d_o does

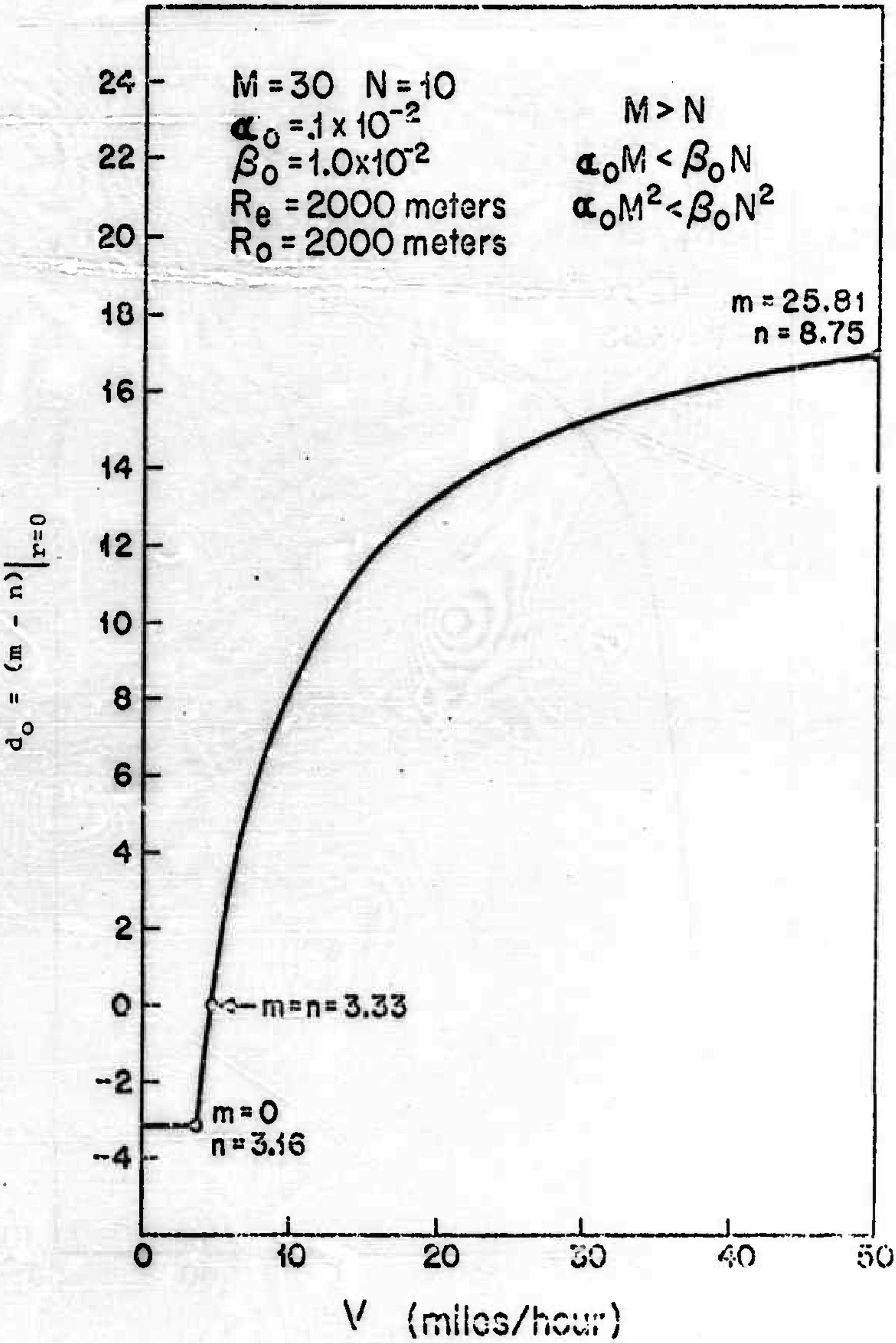

Figure 3 Force Difference at r = 0

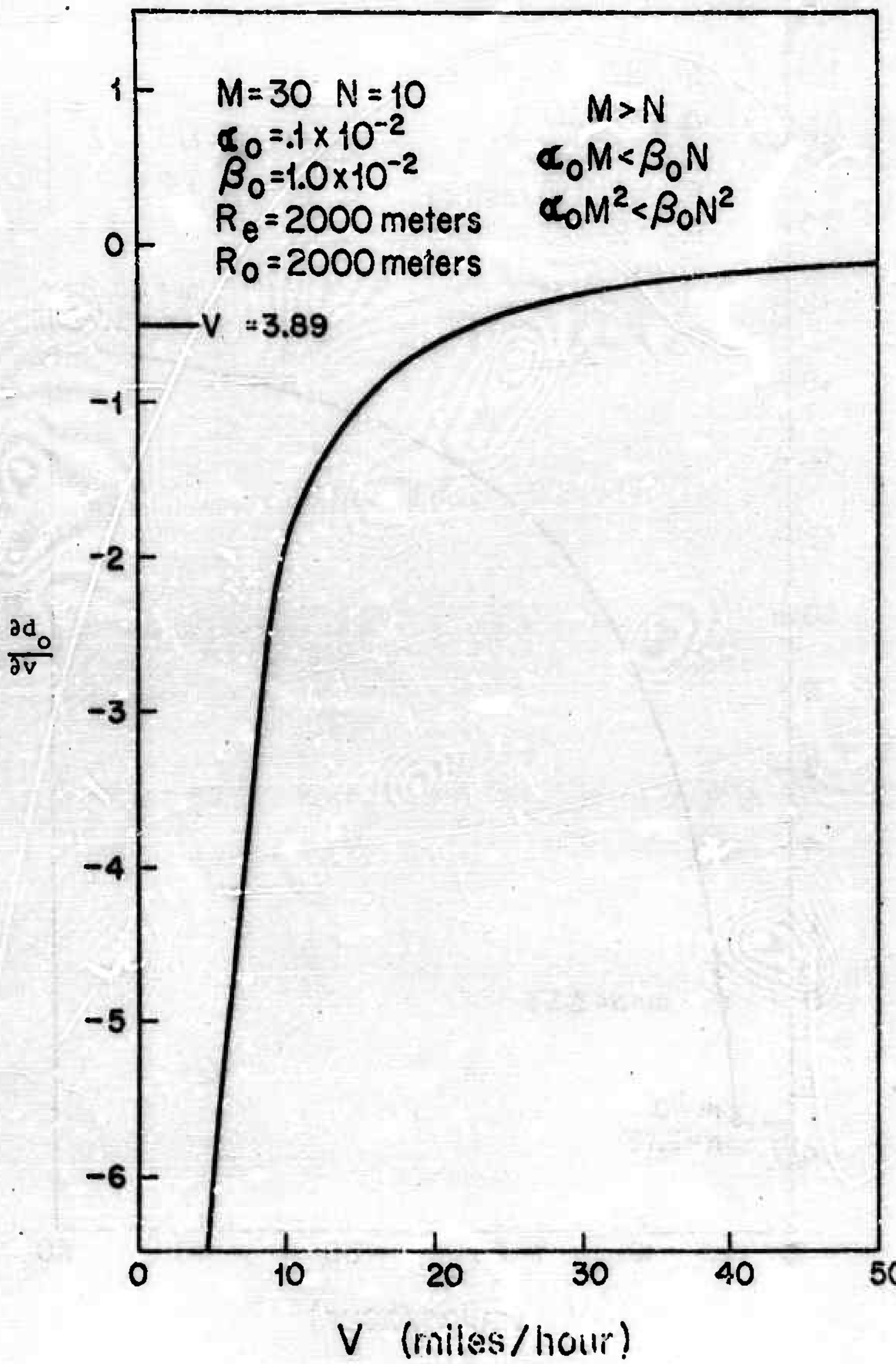

Figure 4 Marginal Effect of Attack Speed on Force Difference at r = 0

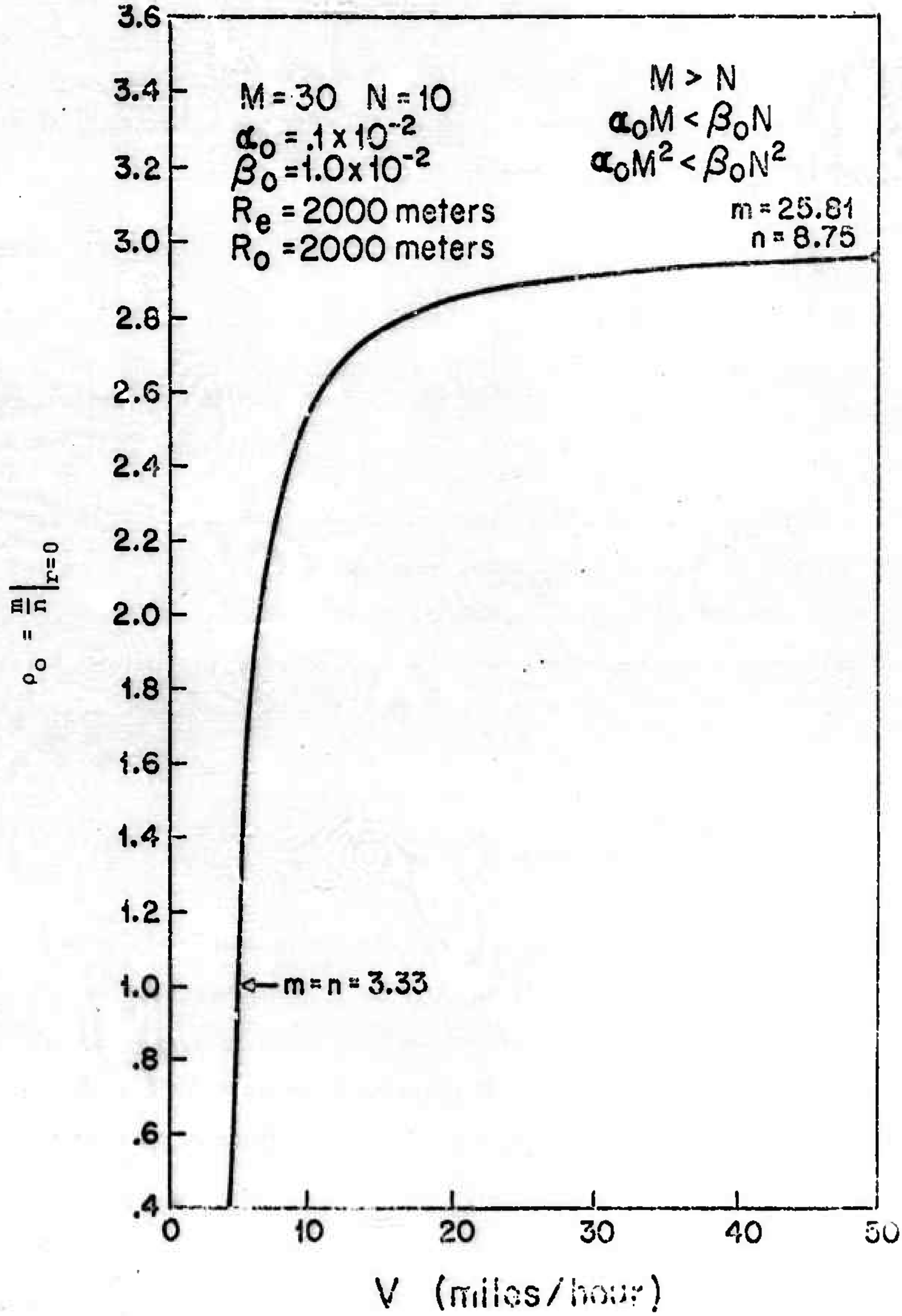

Figure 5 Force Ratio at r = 0

not monotonically decrease to M - N, but has a minimum, after which d_o increases to M - N.

$$d_o' \begin{Bmatrix} <0 \\ =0 \\ >0 \end{Bmatrix} \text{ for } -v \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \frac{C}{\tanh^{-1}\left[\dfrac{\beta_o N - \alpha_o M}{(M - N)\sqrt{\alpha_o \beta_o}}\right]} .$$

Figures 6, 7, and 8 are graphs of d_o, d_o', and ρ_o, respectively, in this situation. Note the minimum of d_o at v = -43.20 and the zero of d_o' at that speed which indicates that the sign of the return changes from negative to positive at this point.

Cases 7 and 8 $\quad M - N < 0$

$$(7)\ \alpha_o M - \beta_o N < 0$$

$$(8)\ \alpha_o M - \beta_o N = 0 .$$

In both situations Red has quadratic superiority, and hence, for $-v \leq -v^{m=0}$, Blue will be annihilated at some $r \geq 0$. Further, since $d_o < 0$ for $-v > -v^{m=0}$, Blue is never terminally superior. However, $d_o' < 0$ and thus this inferiority at r = 0 decreases as speed increases, to M - N. Minimum d_o (<0) occurs at $-v \leq -v^{m=0}$.

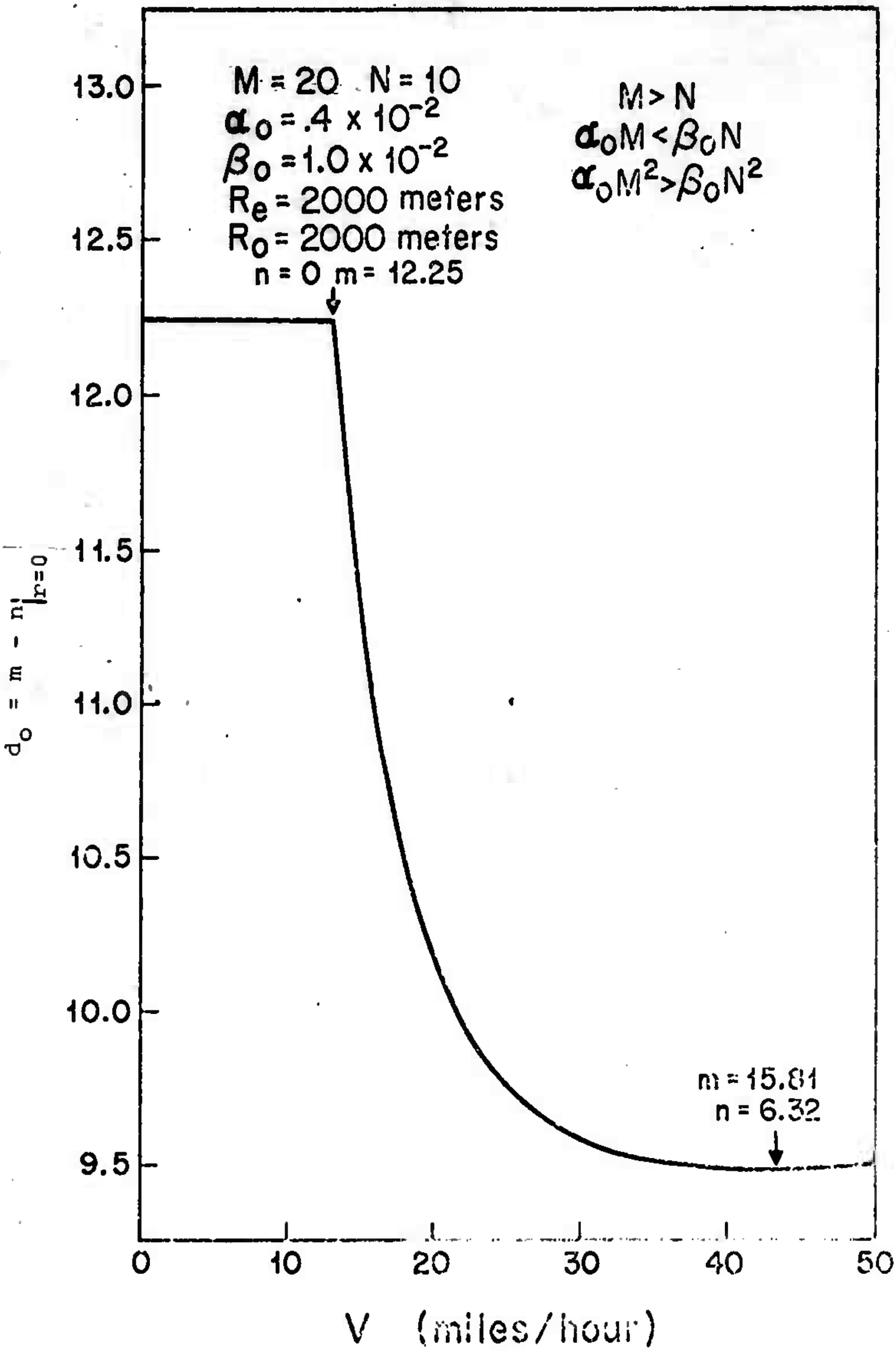

Figure 6 Force Difference at r = 0

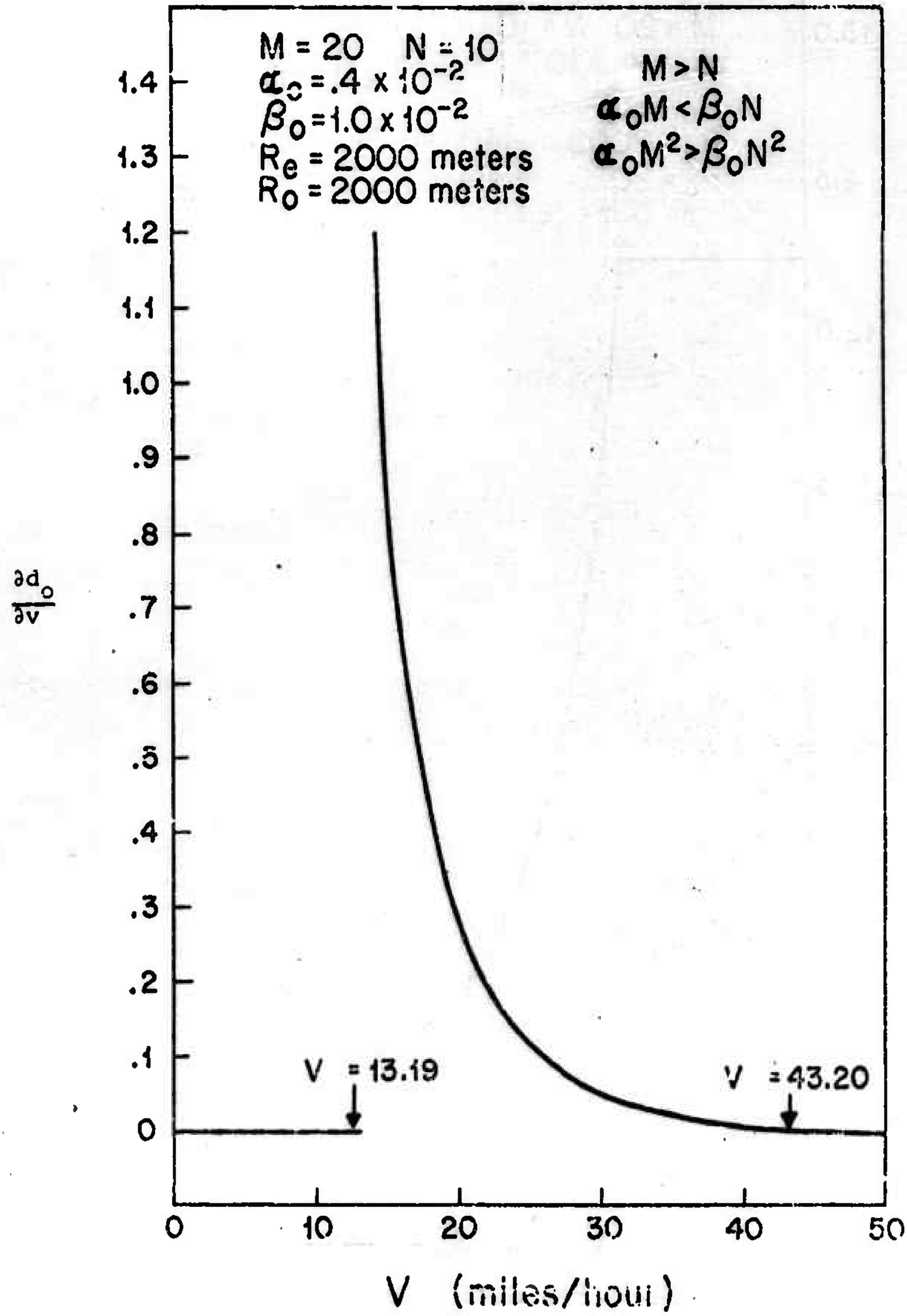

Figure 7 Marginal Effect of Attack Speed on Force Difference at r = 0

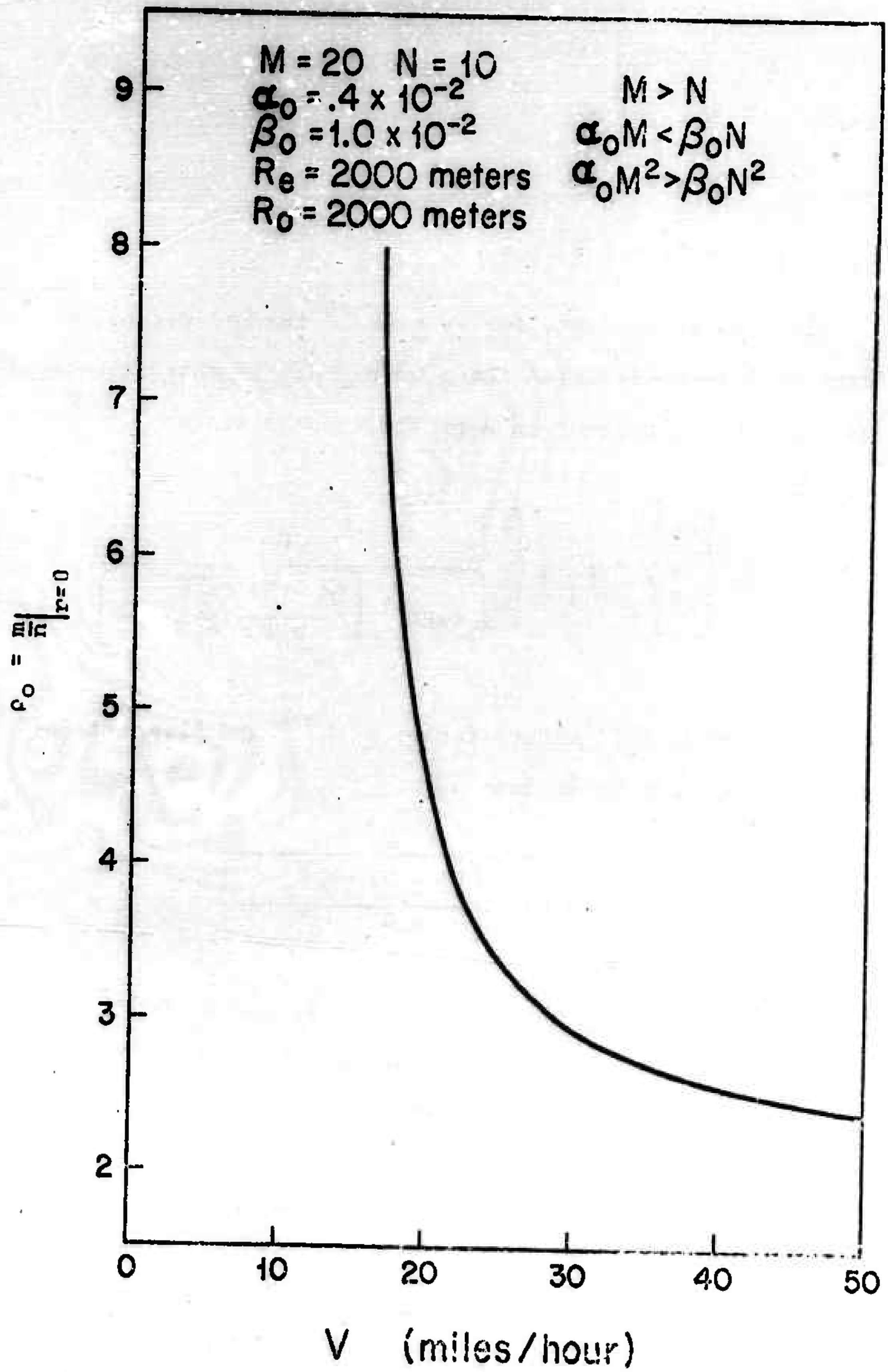

Figure 8 Force Ratio at r = 0

Case 9 $M - N < 0$

$$\alpha_o M - \beta_o N > 0$$

$$\alpha_o M^2 - \beta_o N^2 > 0 .$$

In this engagement, for $-v \leq -v^{n=0}$ the Red defensive force will be annihilated for some $r \geq 0$. In this case terminal superiority depends on a critical speed since

$$d_o \begin{Bmatrix} <0 \\ 0 \\ >0 \end{Bmatrix} \text{ for } -v \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \frac{C}{\tanh^{-1}\left[\dfrac{(M - N)\sqrt{\alpha_o \beta_o}}{\beta_o N - \alpha_o M}\right]} .$$

Here maximum d_o (>0) occurs for $-v \leq -v^{n=0}$ and Blue retains terminal superiority so long as

$$-v < \frac{C}{\tanh^{-1}\left[\dfrac{(M - N)\sqrt{\alpha_o \beta_o}}{\beta_o N - \alpha_o M}\right]} .$$

As speed increases d_o decreases to $M - N$, and $d_o' > 0$. Figures 9, 10, 11 depict d_o, d_o', and ρ_o, respectively, for this case. Note that in this situation it is to the attacker's advantage to proceed slowly.

Case 10 $M - N < 0$

$$\alpha_o M - \beta_o N > 0$$

$$\alpha_o M^2 - \beta_o N^2 < 0 .$$

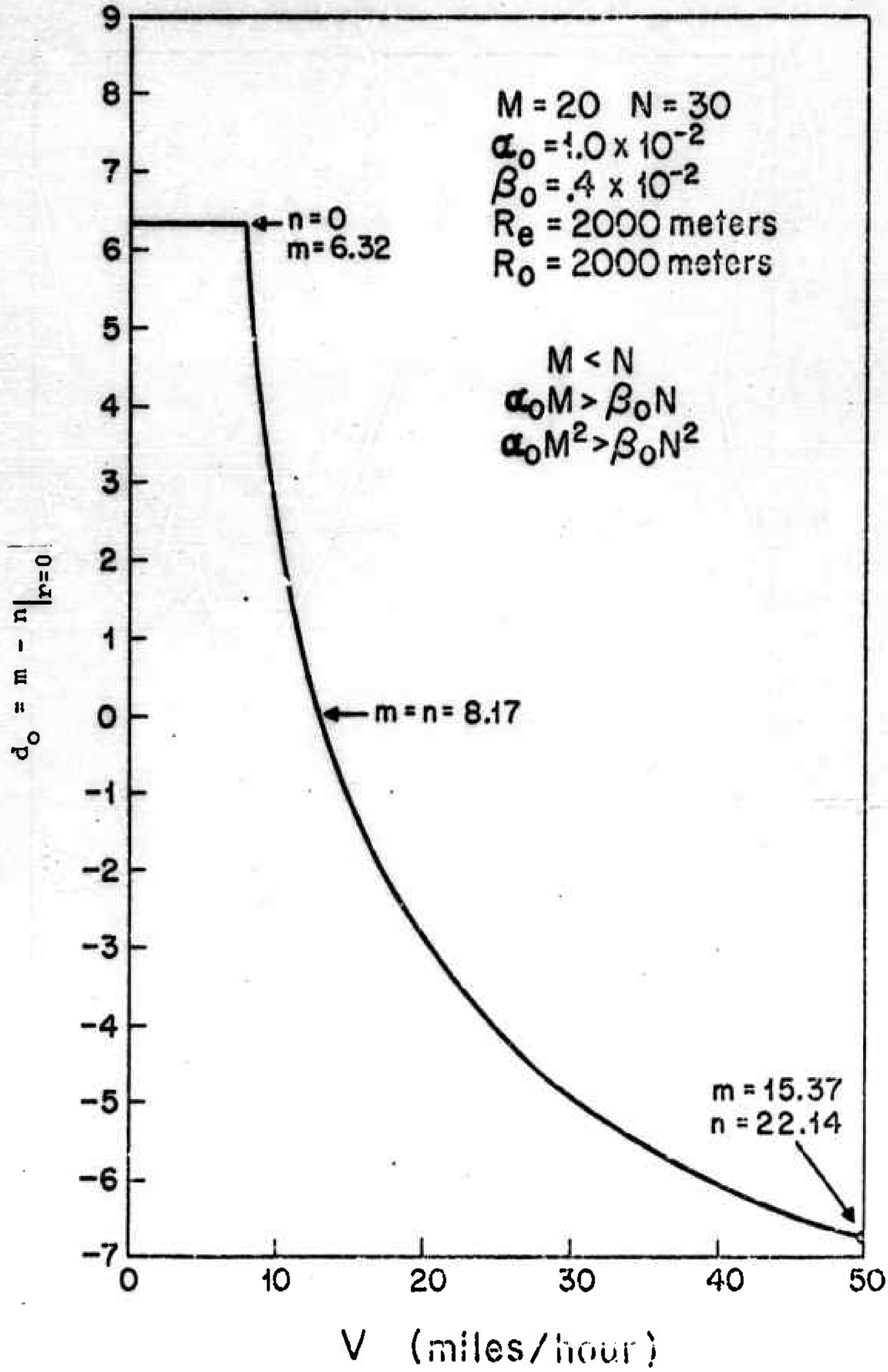

Figure 9 Force Difference at r = 0

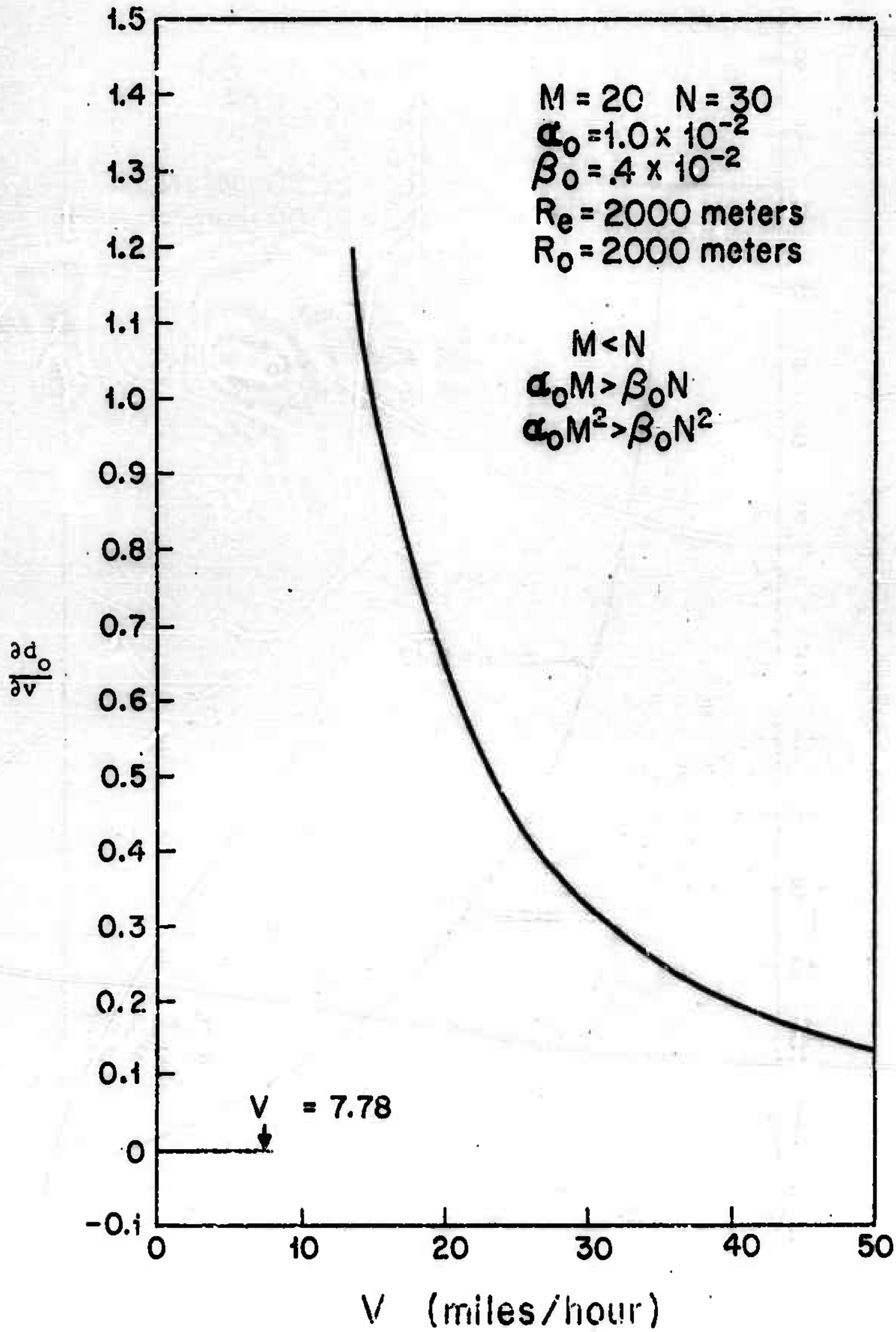

Figure 10 Marginal Effect of Attack Speed on Force Difference at r = 0

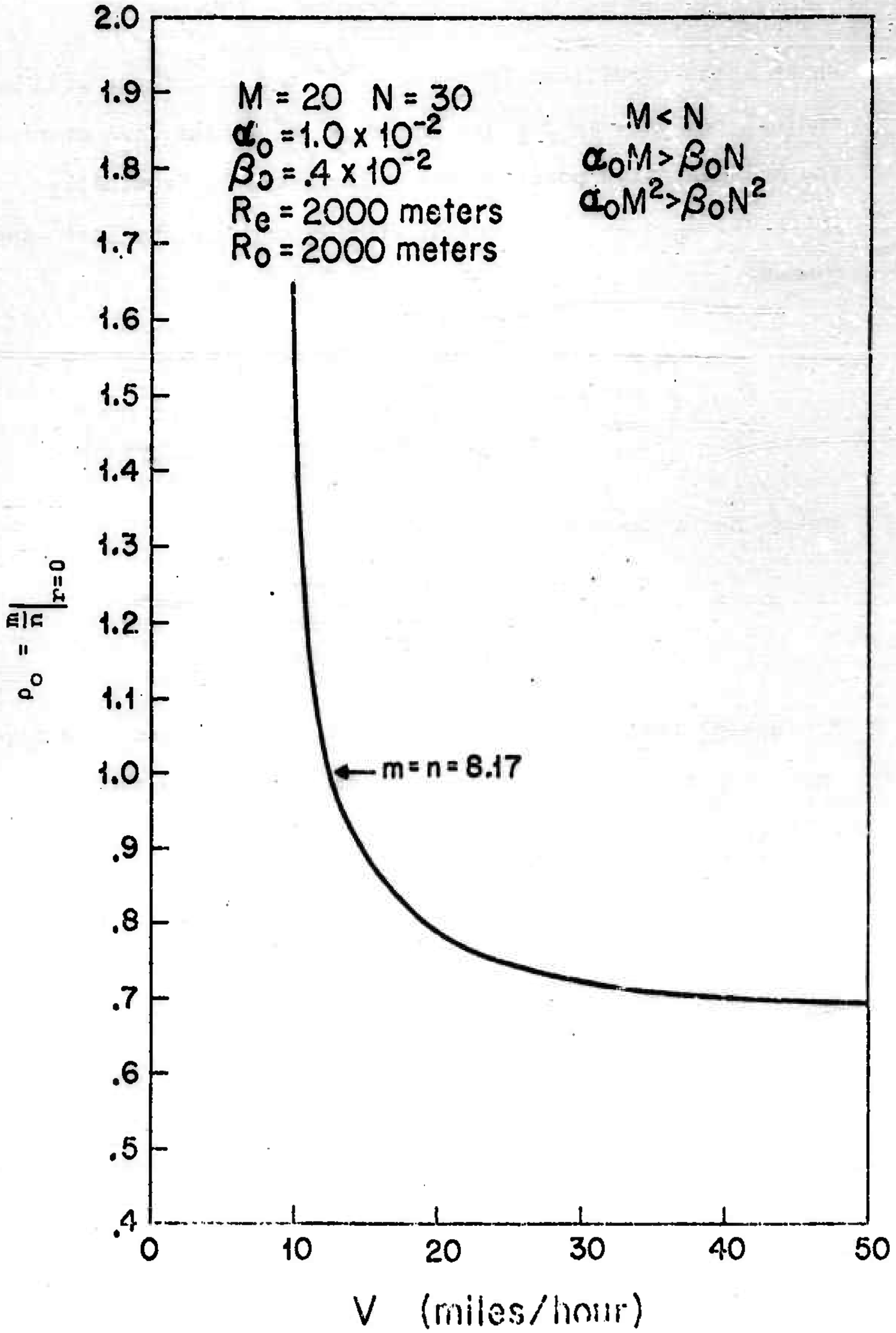

Figure 11 Force Ratio at r = 0

Under these conditions for $-v \leq -v^{m=0}$ the Blue force will be annihilated at some $r \geq 0$. For $-v > -v^{m=0}$ Blue will overrun the Red defensive position but will always be terminally inferior, $d_o < 0$. However, d_o' depends on v in the following manner:

$$d_o' \begin{Bmatrix} <0 \\ 0 \\ >0 \end{Bmatrix} \text{ for } -v \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} \frac{C}{\tanh^{-1}\left[\dfrac{\beta_o N - \alpha_o M}{(M - N)\sqrt{\alpha_o \beta_o}}\right]}$$

and d_o has a maximum at

$$-v = \frac{C}{\tanh^{-1}\left[\dfrac{\beta_o N - \alpha_o M}{(M - N)\sqrt{\alpha_o \beta_o}}\right]} .$$

Minimum d_o (<0) occurs at $-v \leq -v^{m=0}$, d_o increases to a maximum then decreases to M - N. These results are shown in Figures 12, 13, and 14.

Cases 11-13 $\alpha_o M^2 - \beta_o N^2 = 0$.

It was shown in Section 3.3 that in this case

$$d_o = (M - N)\, e^{\theta^o}$$

and

$$d_o = (M - N)\, e^{\theta^c} \frac{\partial \theta^o}{\partial vm} .$$

Hence, if Blue has initial superiority, Blue has terminal superiority and as speed increases, d_o increases to M - N.

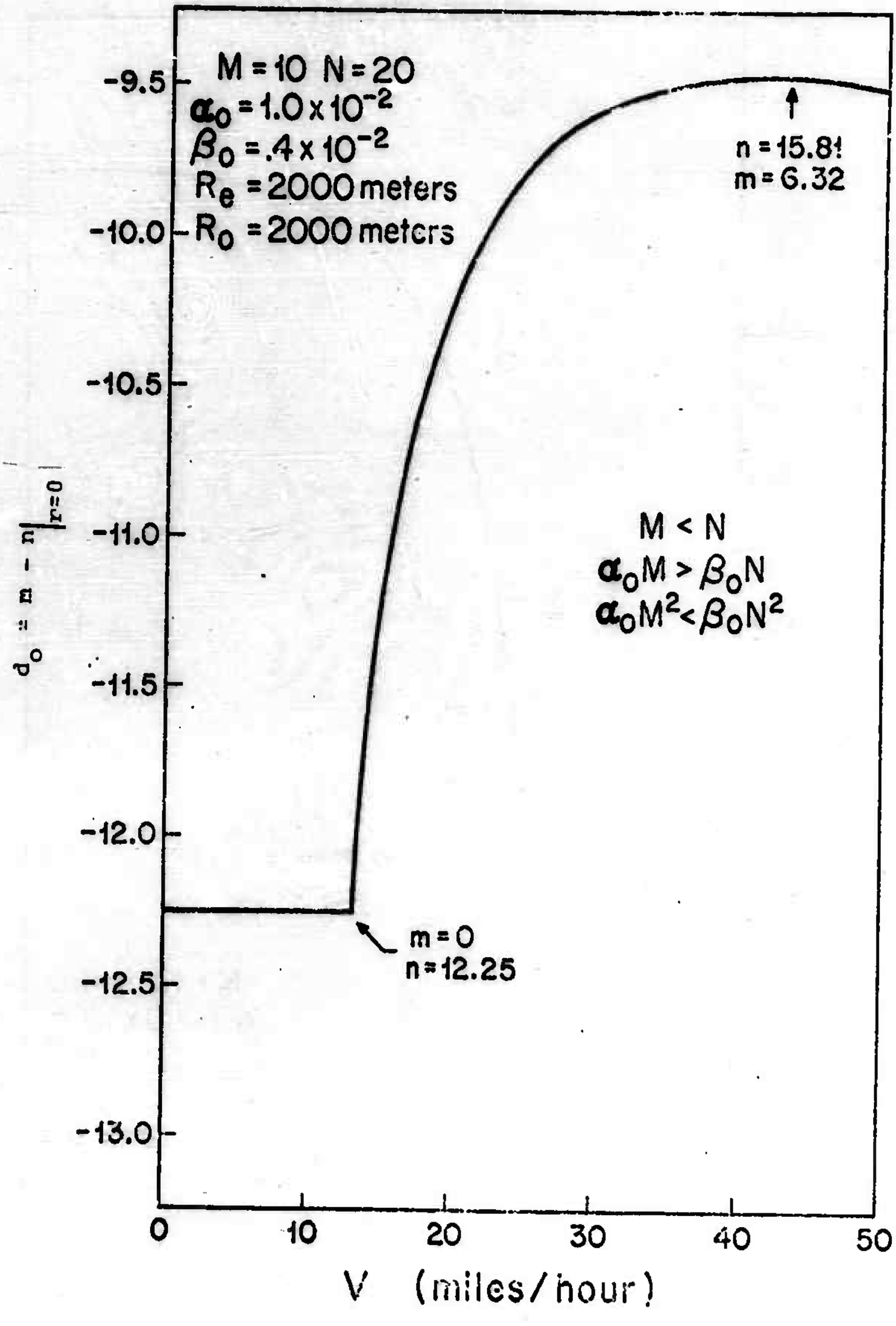

Figure 12 Force Difference at r = 0

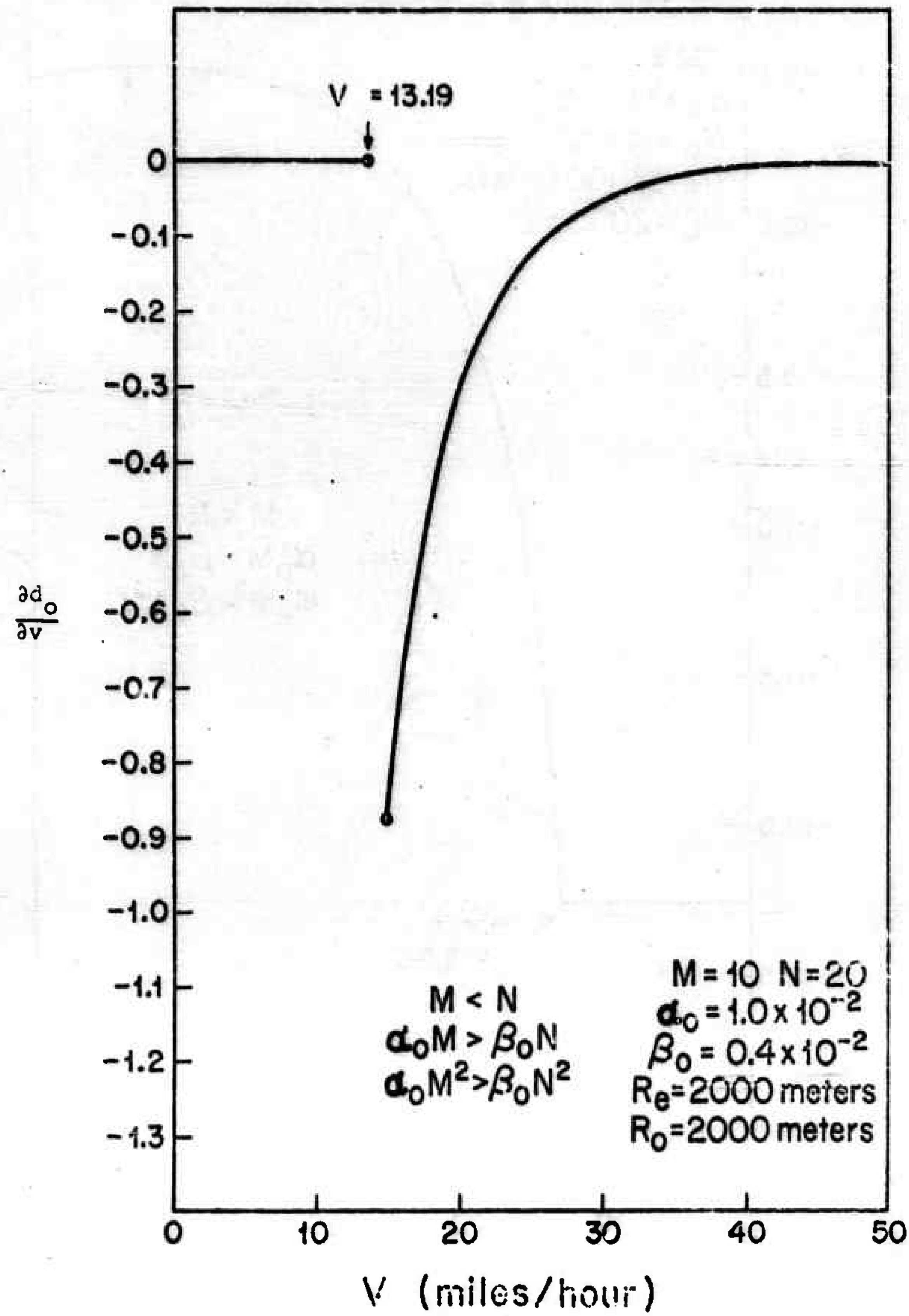

Figure 13 Marginal Effect of Attack Speed on Force Difference at r = 0

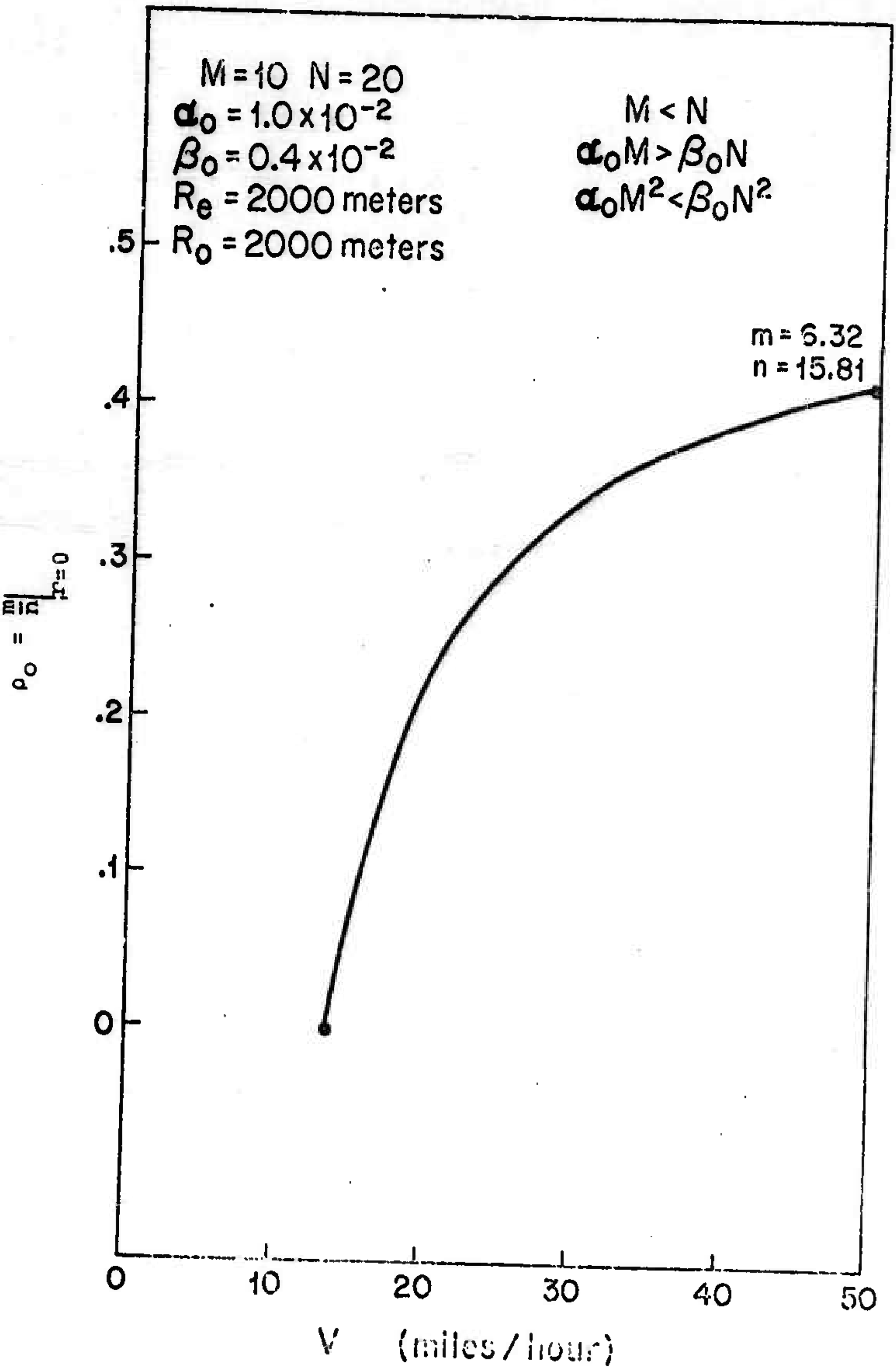

Figure 14 Force Ratio at r = 0

Similarly, if Blue is initially inferior, Blue will not have terminal superiority, $d_o < 0$ and as speed increases d_o decreases to M - N. If M = N, the forces are equal for all assault speeds.

3.5 References

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Weiss, H.K., "Lanchester-Type Models of Warfare," *Proceedings of the First International Conference on Operational Research*, Operations Research Society of America, Baltimore, Maryland, 1957.

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Appendix C, 3

THE LIMIT OF $d_o' = \left.\frac{\partial(m - n)}{\partial v}\right|_{r = 0}$ WHEN SPEED APPROACHES INFINITY

Peter Cherry

In Section 3.4 it was shown that

$$d_o' = \left\{(M - N)\sinh\theta + \frac{\beta_o N - \alpha_o M}{\sqrt{\beta_o \alpha_o}}\cosh\theta\right\}\left(\frac{-c}{v^2}\right).$$

Since $\theta \to 0$ as $v \to -\infty$, the limit of d_o' is zero and the "return" from an increase in attack speed eventually must be diminishing. The "return" is considered as the magnitude of the change Δd_o. This is seen if we write d_o' in the following forms:

$$\text{If } \alpha_o M^2(\beta_o - \alpha_o) > \beta_o N^2(\beta_o - \alpha_o) \equiv \left[(M - N)^2 > \left[\frac{\beta_o N - \alpha_o M}{\sqrt{\alpha_o \beta_o}}\right]^2\right],$$

then

$$d_o' = D\sinh(\theta^o + \phi)\frac{\partial\theta}{\partial v},$$

where[1]

$$D = [\text{sign}\ (M - N)]\sqrt{A^2 - B^2}$$

$$A = M - N$$

$$B = \frac{\beta_o N - \alpha_o M}{\sqrt{\alpha_c \beta_c}}$$

---

[1]The function sign $(x) = \frac{x}{|x|}$.

$$\phi = \tanh^{-1}\left[\frac{\beta_o N - \alpha_o M}{\sqrt{\alpha_o\beta_o}\quad (M - N)}\right].$$

If $(M - N)^2 < \left(\dfrac{\beta N - \alpha_o M}{\sqrt{\alpha_o\beta_o}}\right)^2 \equiv \alpha_o M^2(\beta_o - \alpha_o) < \beta_o N^2(\beta_o - \alpha_o)$,

then

$$d_o' = R \cosh\,(\theta^o + \psi)\,\frac{\partial\theta}{\partial v}\,,$$

where

$$R = [\mathrm{sign}\,(\beta_o N - \alpha_o M)]\sqrt{B^2 - A^2}$$

$$A = M - N$$

$$B = \frac{\beta_o N - \alpha_o M}{\sqrt{\alpha_o\beta_o}}$$

$$\psi = \tanh^{-1}\left[\frac{(M - N)\sqrt{\alpha_o\beta_o}}{\beta_o N - \alpha_o M}\right].$$

If ψ or ϕ is negative, then the functions cosh $(\theta^{o} + \psi)$ and sinh $(\theta^{o} + \phi)$ are monotonic, decreasing as $|v|$ increases and, hence, $|d_o'|$ is monotonic, decreasing as $|v|$ increases.

Letting $f(v) = d_o$ at assault speed v, then

$$|f(v_1) - f(v_2)| = |f'(\xi)||v_1 - v_2| \; .$$

$|f'(\xi)|$ is monotonic, decreasing to zero, and then implies that the "return" $|f(v') - f(v)|$ decreases for $|v|$ increasing if $|v' - v|$ is held constant.

For the cases $\phi > 0$, $\psi > 0$ the above reasoning holds only until $\theta^{o} = -\phi$ or ψ and $f'(\theta) = 0$. In the case $\phi > 0$ we can argue that at $\theta^{o} = -\phi$, the function has a minimum or maximum and from that point increases or decreases to limit M - N as $|v|$ increases. Furthermore, this increase or decrease is monotonic since $d_o = D \cosh(\theta^{o} + \phi)$, $\theta^{o} > -\phi$. Hence, while the "return" from an increase in attack speed may increase, at some point, specifically where $d_o'' = 0$, the return begins to diminish and continues to do so thereafter.

In the case $\psi > 0$, $d_o = R \sinh(\theta^{o} + \psi)$ changes sign at $\theta^{o} = -\psi$. Sinh $(\theta^{o} + \psi)$ is monotonic increasing with respect to θ^{o}. $\theta^{o} = \frac{C}{v}$ is monotonic, increasing with respect to v as $|v|$ increases. The function $\theta^{o} = \frac{C}{v}$, moreover, has a decreasing "return" as $|v|$ increases; hence, sinh $(\theta^{o} + \psi)$ has a decreasing return as $|v|$ increases, i.e., Δd_o decreases, for constant $\Delta|v|$ as $|v|$ increases, and Δd_o does not change

sign in this case.

If $M - N = 0$,

$$d_o' = \frac{\beta_o N - \alpha_o M}{\sqrt{\alpha_o \beta_o}} \cosh\theta \frac{\partial\theta}{\partial v} .$$

If $\beta_o N - \alpha_o M = 0$,

$$d_o' = (M - N) \sinh\theta \frac{\partial\theta}{\partial v} .$$

In both these cases $|f'(\xi)|$ is strictly monotonic, decreasing to zero, and the return $= |\Delta d_o|$ is diminishing with respect to a constant increase in speed.

Chapter 4

VARIABLE ATTRITION RATES, ANALYTICAL RESULTS

Donald Ballou

Chapter 1 of this part of the report considered the case of constant attrition rates for both the Red and Blue weapons. Chapter 2 presented the solution to a special case of variable attrition rates in which their ratio is a constant. The effect of mobility for this latter situation was examined in Chapter 3. In this chapter we consider the general form of the homogeneous-force battle model with variable attrition rates[1]

$$\frac{dn}{dr} = \frac{\alpha(r)}{v(r)}\, m \tag{1}$$

$$\frac{dm}{dr} = \frac{\beta(r)}{v(r)}\, n \tag{2}$$

and the case in which weapons on both sides have linear attrition-rate functions:

$$\alpha(r) = \begin{cases} K_\alpha(R_\alpha - r) & r \le R_\alpha \\ 0 & r > R_\alpha \end{cases} \tag{3}$$

$$\beta(r) = \begin{cases} K_\beta(R_\beta - r) & r \le R_\beta \\ 0 & r > R_\beta \, . \end{cases} \tag{4}$$

[1]Notation used in this chapter corresponds to that employed in previous ones.

In the general case the assault speed v(r) is a positive function of r and in the linear attrition-rate case is assumed constant.

The general methods applied to the study of these differential equations are

(1) generation of a sequence of successive approximations which converge to the solution of the equations, each approximation of which may be generated from the preceding approximation by elementary mathematical operations (some analysis of error bounds is included);

(2) generation of a power series solution to the system of differential equations;

(3) comparison techniques to generate expressions for upper and lower bounds to the solution of the system of equations; and

(4) quasi-linearization to obtain a solution for the ratio $\rho = n/m$ as the maximum of a fairly complex integral and algebraic expression.

Although none of these techniques has led to immediately useful results, they have given rise to some limited insights and show promise for more interesting results with further research.

The next four sections are devoted to presenting the results of these studies. They are presented in the order listed above, which corresponds to our present understanding of their

usefulness and promise. The results in each case are stated with several of the proofs only outlined or omitted where their developments are obvious or mathematically straightforward. A discussion of the different approaches and an evaluation of their relative strengths and weaknesses is presented in Section 4.5. Future research directions are discussed in Section 4.6 along with some thoughts on ways to enrich the present results.

Clearly, applications of the solution functions are meaningless if they are negative. However, from a mathematical point of view negative values for the surviving numbers of forces presents no difficulty, and hence, in all the presentations the functions $n(r)$, $m(r)$ are considered on the closed interval $[0,R_o]$ regardless of their sign.

The following theorem is of a general nature and gives an idea of the behavior of the zeros of the solution $[n(r), m(r)]$ to (1) and (2) on the interval $[0,R_o)$.

Theorem 1

The solution functions $n(r)$ and $m(r)$ to (1) and (2) can vanish at most once on $[0,R_o)$. If either $n(r)$ or $m(r)$ should vanish on $[0,R_o)$, then the other cannot. In particular, $n(r_1) = m(r_1) = 0$, $r_1 \varepsilon [0,R_o)$, is impossible.

The proof is based on arguments concerning the sign of the derivatives of n and m at any zero.

4.1 Method of Successive Approximations

The first part of this section gives results for general $\alpha(r)$ and $\beta(r)$, while the second considers the case of linear $\alpha(r)$ and $\beta(r)$. In order to use the method of successive approximations, the system of equations are rewritten in matrix form:

$$\frac{d\phi}{dr} = A(r)\phi$$
$$\phi(R_o) = \xi \ , \tag{5}$$

where

$$\phi(r) = \begin{pmatrix} n(r) \\ m(r) \end{pmatrix}$$

$$A(r) = \begin{pmatrix} 0 & \alpha(r)/v \\ \beta(r)/v & 0 \end{pmatrix} \tag{6}$$

$$\xi = \begin{pmatrix} N \\ M \end{pmatrix} \ . \tag{7}$$

Note that if ϕ is continuous and satisfies

$$\phi(r) = \xi + \int_{R_o}^{r} A(s)\phi(s)\, ds \ , \tag{8}$$

then, since A is continuous,

$$\frac{d\phi}{dr}(r) = A(r)\phi(r) \ .$$

The derivative

$$\frac{d\phi}{dr} = \begin{pmatrix} dn/dr \\ dm/dr \end{pmatrix}$$

and

$$\int_{R_o}^{r} \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \end{pmatrix} ds = \begin{pmatrix} \int_{R_o}^{r} \psi_1(s)\, ds \\ \int_{R_o}^{r} \psi_2(s)\, ds \end{pmatrix} ,$$

with $\psi(s) = A(s)\phi(s)$. Thus a solution to our original equations is a ϕ satisfying (8).

The solution ϕ may be obtained using the method of successive approximations. For this let

$$\phi_o(r) = \xi , \tag{9}$$

and define $\phi_j(r)$ recursively by

$$\phi_{j+1}(r) = \xi + \int_{R_o}^{r} A(s)\phi_j(s)\, ds , \quad j = 0,1,2,\ldots. \tag{10}$$

The following lemma is the key step in showing that the sequence $\left\{\phi_j\right\}_{j=0}^{\infty}$ converges to a solution ϕ.

Lemma: Let

$$K(r) = \int_{r}^{R_o} \left[\frac{\alpha(s)}{v} + \frac{\beta(s)}{v}\right] ds, \quad 0 \leq r \leq R_o . \tag{11}$$

then

$$\left\|\phi_j(r) - \phi_{j-1}(r)\right\| \leq \left\|\xi\right\| \frac{|K(r)|^j}{j!}, \qquad j = 1,2,3,\ldots,$$

where

$$\left\|\vec{a}\right\| = \left\|(a_1,a_2)\right\| = |a_1| + |a_2| \quad .$$

The lemma is proved by fairly straightforward induction, using integration by parts.

Theorem 2

The approximations ϕ_k given by (10) converge uniformly to the solution ϕ of (8) in the norm given by

$$\left\|\vec{a}\right\| = \left\|(a_1,a_2)\right\| = |a_1| + |a_2| \quad .$$

That is, given $\varepsilon > 0$, there exists a k such that

$$|n_k(r) - n(r)| + |m_k(r) - m(r)| < \varepsilon , \quad 0 \leq r \leq R_o ,$$

where $\phi(r) = (n(r),m(r))$ solves (1) and (2). Furthermore,

$$\left\|\phi_j(r) - \phi(r)\right\| < E_j(r) \quad , \tag{12}$$

where

$$E_j(r) = (N + M)\left[\exp K(r) - \sum_{\ell=0}^{j} \frac{[K(r)]^\ell}{\ell!}\right], \tag{13}$$

with $K(r)$ given by (11).

This is a direct consequence of the lemma, using the power series expansion of e^x and the Cauchy convergence criterion.

Theorem 3

For each j, the maximum value on $[0,R_o]$ for the error bound (E_j) on the approximation ϕ_j occurs at $r = 0$. Furthermore, $E_j(r)$ increases as r decreases. This observation is proved from the positiveness of α, β, and v.

The approximations $\phi_k(r)$ are most easily expressed in terms of the following quantities:

$$I_1(r) = \int_{R_o}^{r} \frac{\alpha(s)}{v}\,ds \;; \qquad I_2(r) = \int_{R_o}^{r} \frac{\beta(s)}{v}\,ds \;;$$

$$I_{12}(r) = \int_{R_o}^{r} \frac{\alpha(s)}{v} I_2(s)\,ds; \qquad I_{21}(r) = \int_{R_o}^{r} \frac{\beta(s)}{v} I_1(s)\,ds \;;$$

$$I_{121}(r) = \int_{R_o}^{r} \frac{\alpha(s)}{v} I_{21}(s)\,ds \;; \qquad I_{212}(r) = \int_{R_o}^{r} \frac{\beta(s)}{v} I_{12}(s)\,ds \;;$$

$$I_{1212}(r) = \int_{R_o}^{r} \frac{\alpha(s)}{v} I_{212}(s)\,ds \;; \qquad I_{2121}(r) = \int_{R_o}^{r} \frac{\beta(s)}{v} I_{121}(s)\,ds \;;$$

....

It should be clear how to define $I_{(\ldots)}$ for any sequence of 1's and 2's with the 1's and 2's alternating.

Remark: Since $\alpha(r)$, $\beta(r)$, $v > 0$, and since $r < R_o$,

$$I_1(r),\ I_2(r),\ I_{121}(r),\ I_{212}(r),\ \ldots < 0\ ,$$

$$I_{12}(r),\ I_{21}(r),\ I_{1212}(r),\ I_{2121}(r),\ \ldots > 0\ .$$

Theorem 4

The approximation $\phi_k(r)$ as given by (10) has the form

$$\phi_k(r) = \begin{pmatrix} n_k(r) \\ m_k(r) \end{pmatrix} = \begin{pmatrix} N + MI_1(r) + \ldots + C_k I_{s_k}(r) \\ M + NI_2(r) + \ldots + D_k I_{t_k}(r) \end{pmatrix}, \tag{14}$$

where

$$C_k = \begin{cases} N, & k \text{ even} \\ M, & k \text{ odd} \end{cases} ;$$

$$D_k = \begin{cases} N, & k \text{ odd} \\ M, & k \text{ even} \end{cases} ;$$

$$s_k = \begin{cases} 121 \ldots 121 & k \text{ odd} \\ \quad k\text{-integers} & \\ 121 \ldots 212 & k \text{ even} \end{cases} ;$$

$$t_k = \begin{cases} 212 \ldots 212 & k \text{ odd} \\ \quad k\text{-integers} & \\ 212 \ldots 121 & k \text{ even} \end{cases} .$$

The proof of this transformation is by straightforward induction.

Remark: The first several approximations are given below explicitly. The alternating nature of the approximations is emphasized by introducing the absolute value of the integrals:

$$n_1(r) = N - M\ |I_1(r)|$$
$$m_2(r) = M - N\ |I_2(r)|$$

$$n_2(r) = N - M\ |I_1(r)| + N\ |I_{12}(r)|$$
$$m_2(r) = M - N\ |I_2(r)| + M\ |I_{21}(r)|$$

$$n_3(r) = N - M\ |I_1(r)| + N\ |I_{12}(r)| - M\ |I_{121}(r)|$$
$$m_3(r) = M - N\ |I_2(r)| + M\ |I_{21}(r)| - N\ |I_{212}(r)|\ .$$

A restatement of theorem 2 in these terms is as follows:

Theorem 5

The solutions n(r) and m(r) to (1) and (2) have the alternating series representation:

$$n(r) = N + MI_1(r) + NI_{12}(r) + MI_{121}(r) + NI_{1212}(r) + \ldots$$

$$(15)$$

$$m(r) = M + NI_2(r) + MI_{21}(r) + NI_{212}(r) + MI_{2121}(r) + \ldots.$$

Further theorems which may be obtained by straightforward manipulation include:

(1) *Theorem 6*

If $\int_r^{R_o} \frac{\alpha(s)}{v} < 1$ and $\int_r^{R_o} \frac{\beta(s)}{v}\, ds < 1$,

then for r fixed but arbitrary, $0 \leq r \leq R_o$,

$$|I_1(r)| > |I_{12}(r)| > |I_{121}(r)| > |I_{1212}(r)| > \ldots$$

$$|I_2(r)| > |I_{21}(r)| > |I_{212}(r)| > |I_{2121}(r)| > \ldots .$$

(2) *Theorem 7*

Suppose $\int_0^{R_o} \frac{\alpha(s)}{v}\, ds < 1$ and $\int_0^{R_o} \frac{\beta(s)}{v}\, ds < 1$.

If M = N, then any of the following conditions guarantees that n(r) vanishes on $[0,R_o)$:

(a) $1 + I_1(0) + I_{12}(0) \leq 0$;

(b) $1 + I_1(0) + I_{12}(0) + I_{121}(0) + I_{1212}(0) \leq 0$;

(c) $1 + I_1(0) + I_{12}(0) + I_{121}(0) + I_{1212}(0) + I_{12121}(0)$

$+ I_{121212}(0) \leq 0$.

Remark: The above conditions (a), (b), (c) get successively weaker, i.e., (a) implies (b) but (b) does not necessarily imply (a), etc.

(3) *Theorem 8*

Let λ, $0 < \lambda \leq 1$, be specified, and let r_1, $0 \leq r_1 < R_o$, be given. If the parameters $\alpha(r)$, $\beta(r)$, N, M, v are chosen in such a way that

$$n_j(r_1) = \lambda m_j(r_1) , \tag{16}$$

then

$$|n(r_1) - \lambda m(r_1)| < E_j(r_1) ,$$

where $E_j(r)$ is given by (13) and $(n(r), m(r))$ is the solution to (1) and (2).

(4) *Theorem 9*

Let $\lambda > 0$ be given and fix $r \varepsilon [0, R_o]$. Then $n(r) = \lambda m(r)$ if, and only if,

$$N \Bigg[1 - \frac{\lambda}{v} \int_{R_o}^{r} \beta(s)ds + \frac{1}{v^2} \int_{R_o}^{r} \int_{R_o}^{s_2} \alpha(s_2)\beta(s_1)ds_1 ds_2$$

$$- \frac{\lambda}{v^3} \int_{R_o}^{r} \int_{R_o}^{s_3} \int_{R_o}^{s_2} \beta(s_3)\alpha(s_2)\beta(s_1)ds_1 ds_2 ds_3$$

$$+ \frac{1}{v^4} \int_{R_o}^{r} \int_{R_o}^{s_4} \int_{R_o}^{s_3} \int_{R_o}^{s_2} \alpha(s_4)\beta(s_3)\alpha(s_2)\beta(s_1)ds_1 ds_2 ds_3 ds_4$$

$$+ \ldots \Bigg]$$

$$= M\left[\lambda - \frac{1}{v}\int_{R_o}^{r} \alpha(s)\,ds + \frac{\lambda}{v^2}\int_{R_o}^{r}\int_{R_o}^{s_2} \beta(s_2)\alpha(s_1)ds_1 ds_2\right.$$

$$- \frac{1}{v^3}\int_{R_o}^{r}\int_{R_o}^{s_3}\int_{R_o}^{s_2} \alpha(s_3)\beta(s_2)\alpha(s_1)ds_1 ds_2 ds_3$$

$$+ \frac{\lambda}{v^4}\int_{R_o}^{r}\int_{R_o}^{s_4}\int_{R_o}^{s_3}\int_{R_o}^{s_2} \beta(s_4)\alpha(s_3)\beta(s_2)\alpha(s_1)ds_1 ds_2 ds_3 ds_4$$

$$\left. + \ldots\right] .$$

For the remainder of this section we shall consider the linear attrition-rate functions given by (3) and (4) such that $R_\alpha < R_\beta$. To simplify the approximations, we shall later set $R_o = R_\alpha$.

In order to simplify the calculations, perform the transformation

$$r \to \tilde{r} = r - R_o \quad .$$

Under this transformation equation 10, which recursively defines the approximations, assumes the form

$$\begin{pmatrix} n_k(\tilde{r}) \\ m_k(\tilde{r}) \end{pmatrix} = \begin{pmatrix} N \\ M \end{pmatrix} + \int_0^{\tilde{r}} \begin{pmatrix} 0 & K_\alpha(\tilde{R}_\alpha - s)/v \\ K_\beta(\tilde{R}_\beta - s)/v & 0 \end{pmatrix} \begin{pmatrix} n_{k-1}(s) \\ M_{k-1}(s) \end{pmatrix} ds ,$$

where $-R_o \le \tilde{r} \le 0$, $\tilde{R}_\alpha = R_\alpha - R_o$, $\tilde{R}_\beta = R_\beta - R_o$. The approximations $n_1(r)$, $n_2(r)$,...,$n_6(r)$ are given explicitly in Appendix C, 4,1. They are obtained from $n_k(\tilde{r})$ by replacing $\tilde{r}$ with $r - R_o$. It is easily seen that $m_k(r)$ is obtained from $n_k(r)$ by replacing in $n_k(r)$, N with M , M with N, $\tilde{R}_\alpha$ with $\tilde{R}_\beta$, $\tilde{R}_\beta$ with $\tilde{R}_\alpha$, K_α with K_β, K_β with K_α .

From theorem 2, we have two theorems:

Theorem 10

Let the parameters R_α, R_β, K_α, etc., be such that any one of the following conditions is satisfied. Then $n(r)$ must vanish on $[0,R_o]$.

$$\begin{aligned} n_1(0) + E_1(0) &\le 0 , \\ n_2(0) + E_2(0) &\le 0 , \\ n_3(0) + E_3(0) &\le 0 . \end{aligned} \tag{17}$$

Theorem 11

Let the parameters R_α, R_β, K_α, etc., be chosen so that any one of the following conditions is satisfied. Then $n(r)$ cannot vanish on $[0,R_o]$.

$$\begin{aligned} n_1(0) - E_1(0) &\ge 0 , \\ n_2(0) - E_2(0) &\ge 0 , \\ n_3(0) - E_3(0) &\ge 0 , \end{aligned} \tag{18}$$

where

$$E_k(r) = (N + M)\left[\exp K(r) - \sum_{\ell=0}^{k} \frac{[K(r)]^\ell}{\ell!}\right] \tag{19}$$

and

$$K(r) = \frac{1}{2v}(R_o - r)\Big[K_\alpha(2R_\alpha - r - R_o) + K_\beta(2R_\beta - r - R_o)\Big] . \tag{20}$$

The functions $n_k(r)$ are found in Appendix C, 4, 1.

The equations that follow give conditions under which for specified $\lambda > 0$,

$$n_2(0) = \lambda m_2(0) ,$$

$$n_4(0) = \lambda m_4(0) ,$$

$$n_6(0) = \lambda m_6(0) ,$$

in the case $R_o = R_\alpha$ (which implies that $\tilde{R}_\alpha = 0$ and $\tilde{R}_\beta = R_\beta - R_\alpha$). These conditions are given below. They are obtained by equating $n_k(r)$ with $\lambda m_k(r)$, setting $r = 0$, and rearranging terms.

Condition A: $n_2(0) = \lambda m_2(0)$. $(\tilde{R}_\beta = R_\beta - R_\alpha)$.

$$N\left[\left(\frac{K_\alpha K_\beta \tilde{R}_\beta R_\alpha^3}{3} + \frac{K_\alpha K_\beta R_\alpha^4}{8}\right) + \lambda\left(K_\beta \tilde{R}_\beta R_\alpha + \frac{K_\beta R_\alpha^2}{2}\right)v + v^2\right]$$

$$= M\left[\lambda\left(\frac{K_\alpha K_\beta \tilde{R}_\beta R_\alpha^3}{6} + \frac{K_\alpha K_\beta R_\alpha^4}{8}\right) + \left(\frac{K_\alpha R_\alpha^2}{2}\right)v + \lambda v^2\right] .$$

Condition 3: $n_4(0) = \lambda m_4(0)$.

$$
\begin{aligned}
N\Bigg[& \left\{ \frac{K_\alpha^2 K_\beta^2 \tilde{R}_\beta^2 R_\alpha^6}{72} + \frac{11 \cdot K_\alpha^2 K_\beta^2 \tilde{R}_\beta R_\alpha^7}{840} + \frac{K_\alpha^2 K_\beta^2 R_\alpha^8}{384} \right\} \\
& + \lambda \left\{ \frac{K_\alpha K_\beta^2 \tilde{R}_\beta^2 R_\alpha^4}{12} + \frac{11 \cdot K_\alpha K_\beta^2 \tilde{R}_\beta R_\alpha^5}{120} + \frac{K_\alpha K_\beta^2 R_\alpha^6}{48} \right\} v \\
& + \left\{ \frac{K_\alpha K_\beta \tilde{R}_\beta R_\alpha^3}{3} + \frac{K_\alpha K_\beta R_\alpha^4}{8} \right\} v^2 \\
& + \lambda \left\{ K_\beta \tilde{R}_\beta R_\alpha + \frac{K_\beta R_\alpha^2}{2} \right\} v^3 + v^4 \Bigg] \\
= M\Bigg[& \lambda \left\{ \frac{K_\alpha^2 K_\beta^2 \tilde{R}_\beta^2 R_\alpha^6}{180} + \frac{13 \cdot K_\alpha^2 K_\beta^2 \tilde{R}_\beta R_\alpha^7}{1680} + \frac{K_\alpha^2 K_\beta^2 R_\alpha^8}{384} \right\} \\
& + \left\{ \frac{K_\alpha^2 K_\beta \tilde{R}_\beta R_\alpha^5}{30} + \frac{K_\alpha^2 K_\beta R_\alpha^6}{48} \right\} v \\
& + \lambda \left\{ \frac{K_\alpha K_\beta \tilde{R}_\beta R_\alpha^3}{6} + \frac{K_\alpha K_\beta R_\alpha^4}{8} \right\} v^2 \\
& + \left\{ \frac{K_\alpha R_\alpha^2}{2} \right\} v^3 + \lambda v^4 \Bigg] .
\end{aligned}
$$

Condition C: $n_6(0) = \lambda m_6(0)$.

$$
\begin{aligned}
N\Bigg[& \left\{ \frac{K_\alpha^3 K_\beta^3 \tilde{R}_\beta^3 R_\alpha^9}{2^3\cdot 3^4\cdot 7} + \frac{17\cdot K_\alpha^3 K_\beta^3 \tilde{R}_\beta^2 R_\alpha^{10}}{2^5\cdot 3^2\cdot 5^2\cdot 7} + \frac{211\cdot K_\alpha^3 K_\beta^3 \tilde{R}_\beta R_\alpha^{11}}{2^7\cdot 3^3\cdot 5\cdot 11\cdot 7} + \frac{K_\alpha^3 K_\beta^3 R_\alpha^{12}}{12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2} \right\} \\
& + \lambda \left\{ \frac{K_\alpha^2 K_\beta^3 \tilde{R}_\beta^3 R_\alpha^7}{2^3\cdot 3^2\cdot 7} + \frac{17\cdot K_\alpha^2 K_\beta^3 \tilde{R}_\beta^2 R_\alpha^8}{2^4\cdot 3^2\cdot 5\cdot 7} + \frac{211\cdot K_\alpha^2 K_\beta^3 \tilde{R}_\beta R_\alpha^9}{2^7\cdot 3^3\cdot 5\cdot 7} + \frac{K_\alpha^2 K_\beta^3 R_\alpha^{10}}{10\cdot 8\cdot 6\cdot 4\cdot 2} \right\} v \\
& + \left\{ \frac{K_\alpha^2 K_\beta^2 \tilde{R}_\beta^2 R_\alpha^6}{2^3\cdot 3^2} + \frac{11\cdot K_\alpha^2 K_\beta^2 \tilde{R}_\beta R_\alpha^7}{2^3\cdot 3\cdot 5} + \frac{K_\alpha^2 K_\beta^2 R_\alpha^8}{8\cdot 6\cdot 4\cdot 2} \right\} v^2 \\
& + \lambda \left\{ \frac{K_\alpha K_\beta^2 \tilde{R}_\beta^2 R_\alpha^4}{2^2\cdot 3} + \frac{11\cdot K_\alpha K_\beta^2 \tilde{R}_\beta R_\alpha^5}{2^3\cdot 3\cdot 5} + \frac{K_\alpha K_\beta^2 R_\alpha^6}{6\cdot 4\cdot 2} \right\} v^3 \\
& + \left\{ \frac{K_\alpha K_\beta \tilde{R}_\beta R_\alpha^3}{3} + \frac{K_\alpha K_\beta R_\alpha^4}{8} \right\} v^4 + \lambda \left\{ K_\beta \tilde{R}_\beta R_\alpha + \frac{K_\beta R_\alpha^2}{2} \right\} v^5 + v^6 \Bigg] \\
= M\Bigg[& \lambda \left\{ \frac{K_\alpha^3 K_\beta^3 \tilde{R}_\beta^3 R_\alpha^9}{2^5\cdot 3^4\cdot 5} + \frac{47\cdot K_\alpha^3 K_\beta^3 \tilde{R}_\beta^2 R_\alpha^{10}}{2^6\cdot 3^3\cdot 5^2\cdot 7} + \frac{271\cdot K_\alpha^3 K_\beta^3 \tilde{R}_\beta R_\alpha^{11}}{2^8\cdot 3^3\cdot 5\cdot 7\cdot 11} + \frac{K_\alpha^3 K_\beta^3 R_\alpha^{12}}{12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2} \right\} \\
& + \left\{ \frac{K_\alpha^3 K_\beta^2 \tilde{R}_\beta^2 R_\alpha^8}{2^5\cdot 3^2\cdot 5} + \frac{13\cdot K_\alpha^3 K_\beta^2 \tilde{R}_\beta R_\alpha^9}{2^4\cdot 3^3\cdot 5\cdot 7} + \frac{K_\alpha^3 K_\beta^2 R_\alpha^{10}}{10\cdot 8\cdot 6\cdot 4\cdot 2} \right\} v \\
& + \lambda \left\{ \frac{K_\alpha^2 K_\beta^2 \tilde{R}_\beta^2 R_\alpha^6}{2^2\cdot 3^2\cdot 5} + \frac{13\cdot K_\alpha^2 K_\beta^2 \tilde{R}_\beta R_\alpha^7}{2^4\cdot 3\cdot 5\cdot 7} + \frac{K_\alpha^2 K_\beta^2 R_\alpha^8}{8\cdot 6\cdot 4\cdot 2} \right\} v^2
\end{aligned}
$$

$$+ \left\{ \frac{K_\alpha^2 K_\beta \hat{R}_\beta R_\alpha^5}{30} + \frac{K_\alpha^2 K_\beta R_\alpha^5}{48} \right\} v^3$$

$$+ \lambda \left\{ \frac{K_\alpha K_\beta \hat{R}_\beta R_\alpha^3}{6} + \frac{K_\alpha K_\beta R_\alpha^4}{8} \right\} v^4 + \left\{ \frac{K_\alpha R_\alpha^2}{2} \right\} v^5 + \lambda\, v^6 \Bigg] .$$

Theorem 12

Let λ satisfy $0 < \lambda \leq 1$. If the parameters K_α, K_β, R_α, R_β, v, N, M are such that

(a) Condition A holds, then

$$|n(0) - \lambda m(0)| < E_2(0) ;$$

(b) Condition B holds, then

$$|n(0) - \lambda m(0)| < E_4(0) ;$$

(c) Condition C holds, then

$$|n(0) - \lambda m(0)| < E_6(0) ,$$

where E_j is defined from (19).

A restatement of theorem 9 gives

Theorem 13

Let $\lambda > 0$ be given, and fix $r \varepsilon [0,R_o)$. Then $n(r) = \lambda m(r)$ if, and only if, the parameters K_α, K_β, R_α, R_β, v, N, M are chosen so that the following holds:

$$\lambda\Bigg[1 + \sum_{j=1}^{\infty} \frac{K_\alpha^j K_\beta^j}{v^{2j}} \int_0^{\tilde r}\int_0^{s_{2j}}\int_0^{s_{2j-1}}\cdots\int_0^{s_2} (\tilde R_\alpha - s_{2j})(\tilde R_\beta - s_{2j-1}) \cdots$$

$$(\tilde R_\alpha - s_2)(\tilde R_\beta - s_1)ds_1 ds_2 \ldots ds_{2j}$$

$$- \lambda\sum_{j=1}^{\infty} \frac{K_\alpha^{j-1} K_\beta^j}{v^{2j-1}} \int_0^{\tilde r}\int_0^{s_{2j-1}}\cdots\int_0^{s_2} (\tilde R_\beta - s_{2j-1})(\tilde R_\alpha - s_{2j-2}) \cdots$$

$$(\tilde R_\beta - s_1)ds_1 ds_2 \ldots ds_{2j-1}\Bigg]$$

$$= M\Bigg[\lambda + \lambda\sum_{j=1}^{\infty} \frac{K_\alpha^j K_\beta^j}{v^{2j}} \int_0^{\tilde r}\int_0^{s_{2j}}\cdots\int_0^{s_2} (\tilde R_\beta - s_{2j})(\tilde R_\alpha - s_{2j-1}) \cdots$$

$$(\tilde R_\beta - s_2)(\tilde R_\alpha - s_1)ds_1 \ldots ds_{2j}$$

$$- \sum_{j=1}^{\infty} \frac{K_\alpha^j K_\beta^{j-1}}{v^{2j-k}} \int_0^{\tilde r}\int_0^{s_{2j-1}}\cdots\int_0^{s_2} (\tilde R_\alpha - s_{2j-1})(\tilde R_\beta - s_{2j-2}) \cdots$$

$$(\tilde R_\alpha - s_1)ds_1 ds_2 \ldots ds_{2j-1}\Bigg] .$$

In the above $\tilde r = r - R_o$, $\tilde R_\alpha = R_\alpha - R_o$, $\tilde R_\beta = R_\beta - R_o$.

4.2 Power-Series Approach

As shown in Chapter 2, consideration of equations 1 and 2 under a constant assault speed ($\omega = \frac{1}{v}\frac{dv}{dr} = 0$) can be combined to produce the following second-order linear differential equation for n(r)

$$\frac{d^2n}{dr^2} - \left(\frac{1}{\alpha}\frac{d\alpha}{dr}\right)\frac{dn}{dr} - \left(\frac{\alpha(r)\beta(r)}{v^2}\right) n = 0 \qquad (21)$$

subject to the initial conditions $n(R_o) = N$ and

$$\frac{dn}{dr}(R_o) = \frac{\alpha(R_o)}{v} M , \qquad (22)$$

where v is a positive constant and $\alpha(r)$ and $\beta(r)$ are non-negative functions on $[0, R_o]$. It is now useful to assume that $\alpha(r)$, $\beta(r) \epsilon C^2[(0,R_o)]$. This Cauchy problem was studied for the linear attrition-rate functions given by (3) and (4).

The specific case studied assumed $R_o = R_\alpha < R_\beta$ and $K_\alpha > K_\beta$. Employing these attrition-rate functions, the Cauchy problem assumes the form

$$\frac{d^2n}{dr^2} + \left(\frac{1}{R_\alpha - r}\right)\frac{dn}{dr} - \frac{K_\alpha K_\beta (R_\alpha - r)(R_\beta - r)}{v^2} n = 0 \qquad (23)$$

$$n(R_o) = N$$

$$\frac{dn}{dr}(R_c) = \frac{K_\alpha (R_\alpha - R_o)}{v} M . \qquad (24)$$

Note that (23) has a singularity at $R_o = R_\alpha$. A solution is found in a neighborhood of $r = R_\alpha$ using the method of Frobenius (Coddington, 1955). The solution obtained by this method has the form

$$n(r) = c_1\phi(r - R_\alpha) + c_2\psi(r - R_\alpha) , \qquad (25)$$

where c_1 and c_2 are determined so that (24) holds.

The functions

$$\phi(r - R_\alpha) = \sum_{j=0}^{\infty} b_j (r - R_\alpha)^{j+2}$$

and

$$\psi(r - R_\alpha) = 1 + \sum_{j=3}^{\infty} a_j (r - R_\alpha)^j .$$

The coefficients b_j are given by

$$b_o = 1; \quad b_1 = 0; \quad b_2 = 0; \quad b_3 = \frac{K_2}{f(5)}$$

$$b_j = \frac{K_2 b_{j-3} - K_1 b_{j-4}}{f(j+2)} , \quad j = 4,5,6,\ldots,$$

where

$$K_1 = K_\alpha K_\beta$$

$$K_2 = K_\alpha K_\beta (R_\alpha - R_\beta)$$

$$f(\lambda) = \lambda^2 - 2\lambda \quad ,$$

while the coefficients a_j are given by

$$a_j = \frac{dg_j}{d\lambda}(0) \ ,$$

where

$$g_1 = 0; \quad g_2 = 0; \quad g_3(\lambda) = K_2\lambda/f(\lambda + 3);$$

$$g_4(\lambda) = \frac{K_1\lambda}{f(\lambda + 4)} \quad ;$$

$$g_j(\lambda) = \frac{K_2 d_{j-3} + K_1 d_{j-4}}{f(\lambda + j)} \ , \quad j = 5,6,7,\ldots.$$

Note that a power-series expansion can be obtained for m(r) following the same procedure. The chief difference is that the expansion will be around R_β rather than R_α .

The Cauchy problem noted above can be converted to a useful dimensionless form by letting

$$y = n/N \tag{26}$$

$$x = m/M \tag{27}$$

$$Y = (R_\alpha - r)\sqrt[4]{K_\alpha K_\beta/v^2} \tag{28}$$

$$R_\Delta = (R_\beta - R_\alpha)\sqrt[4]{K_\alpha K_\beta/v^2} \ . \tag{29}$$

Then

$$\frac{dy}{d\gamma} = \frac{1}{N}\frac{dn}{d\gamma}$$

$$= \frac{1}{N}\frac{dr}{d\gamma}\frac{dn}{dr}$$

$$= \frac{1}{N}\left[-\sqrt[4]{v^2/K_\alpha K_\beta}\right]\left[-\frac{K_\alpha}{v}(R_\alpha - r)m\right] \qquad (30)$$

$$= -\frac{M}{N}\sqrt{K_\alpha/K_\beta}\ \gamma x$$

$$= -\phi\gamma x$$

by letting $\phi = M\sqrt{K_\alpha}/N\sqrt{K_\beta}$.

$$\frac{dx}{d\gamma} = \frac{1}{M}\frac{dm}{d\gamma} = \frac{1}{M}\frac{dr}{d\gamma}\frac{dm}{dr}$$

$$= \frac{1}{M}\left[-\sqrt[4]{\frac{v^2}{K_\alpha K_\beta}}\right]\left[\frac{-K_\beta(R_\beta - r)n}{v}\right]$$

$$= \frac{1}{M}\left[-\sqrt[4]{\frac{v^2}{K_\alpha K_\beta}}\right]\left[\frac{K_\beta(R_\beta - R_\alpha + R_\alpha - r)}{v}\left(\frac{n}{N}\right)N\right]$$

$$= -\frac{N}{M}\sqrt[4]{\frac{v^2}{K_\alpha K_\beta}}\ \frac{K_\beta}{v}\left[\frac{R_\Delta}{\sqrt[4]{K_\alpha K_\beta/v^2}} + \frac{\gamma}{\sqrt[4]{K_\alpha K_\beta/v^2}}\right]y$$

$$= -\frac{N}{M}\left(\frac{v}{\sqrt{K_\alpha K_\beta}}\right)\left(\frac{K_\beta}{v}\right)(R_\Delta + \gamma)\,y$$

$$= -\frac{N}{M}\frac{\sqrt{K_\beta}}{\sqrt{K_\alpha}}(R_\Delta + \gamma)\,y$$

$$= -\frac{1}{\phi}(R_\Delta + \gamma)\,y \quad . \tag{31}$$

Thus, the solution to equations 1 and 2 with a constant assault speed and linear attrition-rate functions can be obtained from the solution of (30) and (31) with initial conditions that, at $\gamma = 0$ ($r = R_o = R_\alpha$), $x = 1$ and $y = 1$. The Cauchy problem is obtained directly from (30) and (31). From (30)

$$-\frac{1}{\phi\gamma}\frac{dy}{d\gamma} = x \ ,$$

which, when substituted in (31), becomes

$$\frac{d\left[\frac{1}{\gamma}\frac{dy}{d\gamma}\right]}{d\gamma} = (R_\Delta + \gamma)\,y \quad .$$

Differentiating again,

$$\frac{1}{\gamma}\frac{d^2y}{d\gamma^2} + \frac{dy}{d\gamma}\left(-\frac{1}{\gamma^2}\right) - (R_\Delta + \gamma)\,y = 0$$

or

$$\frac{d^2y}{d\gamma^2} - \frac{1}{\gamma}\frac{dy}{d\gamma} - \gamma(R_\Delta + \gamma)y = 0 \ , \tag{32}$$

which is readily solved by the method of Frobenius described at the beginning of this section.

We note that the dimensionless parameters R_Δ and ϕ completely characterize the solution and can be used to show the trade-off among relevant parameters. If we assume that N, R_β, and K_β are given and fixed, and if the solution is to remain unchanged (i.e., R_Δ and ϕ are fixed), we must have

$$\sqrt{K_\alpha}\, M = \sqrt{K_\beta}\, N\, \phi$$

and

$$R_\beta = R_\alpha + \sqrt[4]{\frac{v^2}{K_\alpha}}\left(\frac{R_\Delta}{\sqrt[4]{K_\beta}}\right) .$$

Thus we see how M, v, K_α, and R_α can be traded off to obtain a specific final result.

4.3 Comparison Techniques

One method of obtaining information about the solutions n(r) and m(r) to (1) and (2) is to use comparison techniques. To see the principle involved, consider the equations

$$\frac{d^2w}{dr^2} + R(r)w = 0 \quad , \tag{33}$$

$$\frac{d^2z}{dr^2} + Q(r)z = 0 \quad . \tag{34}$$

The following theorem relates solutions of (33) and (34):

Theorem 14

Suppose $w(r)$ is a solution to (33) and $z(r)$ is a solution to (34) which satisfy $w(R_o) \leq z(R_o)$ and $w'(R_o) \geq z'(R_o)$. If $R(r) > Q(r)$ on $[0,R_o]$, then $z(r) > w(r)$, so long as both functions are positive.

In order to apply this theorem first note that the second-order differential equation 21 can be written in the form

$$\frac{d^2n}{dr^2} + a_1(r)\frac{dn}{dr} + a_o(r)n = 0, \tag{35}$$

where

$$a_1(r) = \frac{-1}{\alpha}\frac{d\alpha}{dr}$$

$$a_o(r) = \frac{-\alpha(r)\beta(r)}{v^2} \quad . \tag{36}$$

Perform the transformation

$$w(r) = n(r)\exp\left[\int_{R_o}^{r}\frac{a_1(s)}{2}\,ds\right] \tag{37}$$

to put (35) into the form (33), i.e., equation 35 assumes the form

$$\frac{d^2w}{dr^2} + R(r)w = 0 \ ,$$

where

$$R(r) = a_o(r) - \frac{[a_1(r)]^2}{4} - \frac{1}{2}\frac{da_1(r)}{dr} . \qquad (38)$$

From (37) it follows that n(r) vanishes if, and only if, w(r) vanishes. Thus, to determine where n(r) vanishes, it suffices to determine where w(r) vanishes. But from theorem 14 it is seen that if z(r) is a known function which vanishes on $[0,R_o]$ and satisfies (34) for some function Q(r). Then, a sufficient condition for n(r) to vanish is given by $Q(r) < R(r)$, where R(r) is given by (38). It is desirable to make the difference $R(r) - Q(r)$ as small as possible, for doing this reduces the quantity $z(r) - w(r)$ thus giving better control on the zeros of n(r). It turned out in practice to be difficult to find meaningful conditions using this approach.

Another comparison approach utilizes the ratio[1]

$$\rho(r) = n(r)/m(r) ,$$

which from (1) and (2) satisfies the Riccati equation

$$\frac{d\rho}{dr} = \frac{1}{v}\,[\alpha(r) - \beta(r)\,\rho^2(r)] \qquad (39)$$

for constant assault speed v.

Note that if a known function h(r) satisfies $h(0) = 0$, $h(R_o) = N/M$,

$$h'(P_o) < \rho'(R_o) = \frac{1}{v}\left[\alpha(R_o) - \beta(R_o)\,\frac{N^2}{M^2}\right] ,$$

[1]The reader is cautioned that this ratio is the reciprocal of that used in Chapter 3.

then a sufficient condition for $n(r)$ to vanish is that $\rho(r) < h(r)$, $0 \leq r < R_o$. This approach was carried out for

$$h(r) = \frac{\rho(R_o)}{R_o} r$$

and yielded the following result for linear attrition-rate functions $\alpha(r) = K_\alpha(R_\alpha - r)$, $\beta(r) = K_\beta(R_\beta - r)$.

Theorem 15

Let $R_o < \frac{2}{3} R_\beta$, and let

$$\frac{d\rho}{dr}(R_o) > \frac{\rho(R_o)}{R_o} .$$

Then $n(r)$ vanishes.

4.4 Method of Quasi-Linearization

As mentioned above, the function $\rho(r) = n(r)/m(r)$ satisfies the Riccati equation 39. A solution to (39) is desired which satisfies the initial conditions $\rho(R_o) = N/M$. The method of quasi-linearization obtains a closed-form solution by "linearizing" the ρ^2 term, i.e., by replacing ρ^2 with $\max\limits_{u(r)}(2u\rho - u^2)$, where $u(r)\epsilon C([0,R_o])$.

The following theorem gives a representation for $\rho(r) = n(r)/m(r)$ in the case $\alpha(r)$ and $\beta(r)$ are both linear.

Theorem 16

Let $\alpha(r) = K_\alpha(R_\alpha - r)$ and $\beta(r) = K_\beta(R_\beta - r)$. Then

the initial value problem (39) for $r < R_o$ has the solution

$$\rho(r) = \frac{\max}{u(r)} \left[\frac{N}{M} \exp\left\{ \int_r^{R_o} 2u(\xi)d\xi \right\} \right.$$

$$- \int_r^{R_o} \frac{v}{K_\beta(r_\beta - s)} \left[u^2(s) + \frac{K_\alpha K_\beta}{v^2} (R_\alpha - s)(R_\beta - s) \right]$$

$$\left. \cdot \exp\left\{ \int_r^s 2u(\xi)d\xi \right\} ds \right] . \qquad (40)$$

Corollary

A necessary condition for n to vanish on $[0,R_o]$ is

$$\frac{N}{M} \leq \frac{K_\alpha}{v} \left[R_o \left(R_\alpha - \frac{R_o}{2} \right) \right] .$$

In order to obtain information about $\rho(r)$ as given by (40), it is desirable to have a sequence of approximations.

Theorem 17

Let $\rho(r)$ be the solution to (39). Let

$$h_o(r) = \frac{\beta(r)}{v} \frac{N}{M} ,$$

and define h_n, $n \geq 1$, recursively by

$$h_n' = h_{n-1}^2 - 2h_{n-1} h_n - p(r) h_n - q(r) ,$$

where

$$p(r) = -\frac{\beta'(r)}{\beta(r)}$$

$$q(r) = -\frac{\alpha(r)\beta(r)}{v^2} \quad .$$

Let

$$\rho_n(r) = \frac{v}{\beta(r)} h_n(r) \; .$$

Then for $0 \leq r \leq R_o$,

$$\rho_1(r) \leq \rho_2(r) \leq \dots \leq \rho(r)$$

and

$$\lim_{n\to\infty} \rho_n(r) = \rho(r) \; ,$$

where the convergence is uniform.

4.5 Evaluation of the Different Approaches

In this section the different approaches outlined above are discussed to indicate their respective advantages and disadvantages. By far the most valuable approach to the homogeneous-force model utilizes the method of successive approximations. Except for certain special cases, $\alpha(r)$ and $\beta(r)$ will be such that a series solution to (1) and (2) is readily obtainable. The method of successive approximations yields a series which has the advantage that each additional term is easily derived from the preceding one. The series is such that consecutive terms have alternating signs. Further, there is no need to assume

anything about $\alpha(r)$ and $\beta(r)$ other than continuity. In fact, as is seen from examining the technique described in Section 4.1, it is not necessary to assume that v is a constant, i.e., it is possible to suppose only that v is a non-negative continuous function.

The approximations to $n(r)$ and $m(r)$ are obtained by considering the partial sums. These functions are made especially valuable because of the existence of the error bounds $E_j(r)$. Using the error bounds and the approximations, it is possible to derive conditions under which, for $r_1 \epsilon [0, R_o]$, $n(r_1) = \lambda m(r_1)$, $\lambda > 0$, with a known error bound. Also, conditions are available which guarantee that $n(r)$ vanishes. These conditions can be made as weak as desired, i.e., given any $\epsilon > 0$, a condition can be found which guarantees that $n(r) > 0$ on $[\epsilon, R_o]$ but $n(0) < 0$.

Perhaps the most valuable aspect of this method is that it not only treats general $\alpha(r)$ and $\beta(r)$ but also can be used to study the variable-coefficient heterogeneous-force models. This is because the approach can handle any equation of the form

$$\frac{d\phi}{dr} = A(r)\phi \quad ,$$

where

$$\phi(r) = \begin{pmatrix} \phi_1(r) \\ \cdots\cdot \\ \phi_n(r) \end{pmatrix}$$

and A is a continuous n x n matrix. An error bound is available in this case and is similar in form to that for the homogeneous case.

Finally, the approximations are such that they can be easily programmed. When $\alpha(r)$ and $\beta(r)$ are both linear, there is an algorithm suitable for computer use, which can calculate the n^{th} term in the series from the (n-1) term.

Several charts are given in Appendix C, 4, 2 which give an idea of the accuracy of the various approximations for different values of the parameters when $\alpha(r) = K_\alpha(R_\alpha - r)$ and $\beta(r) = K_\beta(R_\beta - r)$. Using the analog-derived solutions presented in the next chapter, it is seen that interesting behavior of the solution (n(r), m(r)) is encountered for those values of the parameters for which $(n_6(r), m_6(r))$ give a "good" approximation to the solution. Thus the analog solutions can be used to see what values of the parameters are needed to induce significant changes in the behavior of the solution (n(r), m(r)). Then, using the charts found in Appendix C, 4, 2, it is possible to find how many approximations are needed to get an error bound that is sufficiently small.

The most elementary method considered, the power-series technique of Frobenius, is fine from a theoretical point of view in that it gives a solution defined for all r. However, a good error bound for the approximations to n(r) obtained by considering the partial sums is not now available, and any application using the power-series solution would have to work with the

partial sums. Computer tests for the rapidity of convergence are of no use, for it is easy to construct examples of power series that seem to converge rapidly for the first n terms only to diverge eventually. A more serious drawback to this approach lies in the fact that the power series for $m(r)$ is taken about R_β, while that for $n(r)$ is taken about R_α. If $R_\alpha \neq R_\beta$, then it is difficult to compare $n(r)$ and $m(r)$ to obtain conditions under which they are equal, etc. Finally, the solution (25) cannot be easily modified to accommodate other than linear $\alpha(r)$ and $\beta(r)$.

The comparison techniques developed in Section 4.3 have the potential of being quite useful. Using theorem 14, it is possible, in theory at least, to find functions $u_1(r)$ and $\ell_1(r)$ such that

$$\ell_1(r) \leq n(r) \leq u_1(r)$$

and functions $u_2(r)$ and $\ell_2(r)$ such that

$$\ell_2(r) \leq m(r) \leq u_2(r) .$$

If the quantities $u_1(r) - \ell_1(r)$ and $u_2(r) - \ell_2(r)$ are "small," then a very good idea of the behavior of $n(r)$ and $m(r)$ is available. The difficulty, of course, is to determine the functions $\ell_1(r)$, $\ell_2(r)$, $u_1(r)$, $u_2(r)$. Finding them is not easy, for a relationship of the form $R(r) > Q(r)$ must hold on $[0, R_o]$, where the significance of R and Q is given in [4.3]. However, for certain $\alpha(r)$ and $\beta(r)$ finding the comparison functions $u_1(r)$,

etc., might not be too difficult. Otherwise, considerable ingenuity is apparently required to find "good" bounds $u_1(r)$, $u_2(r)$, etc.

It will be recalled that the second comparison technique described in [4.3] utilized the ratio $\rho(r) = n(r)/m(r)$. By working with the "bounding" function

$$h(r) = \frac{\rho(R_o)}{R_o} r \;,$$

a rather strong condition was found which guarantees that $n(r)$ vanishes (theorem 15). In order to obtain weaker conditions, functions of the form

$$h_n(r) = \frac{N}{M}\left(\frac{r}{R_o}\right)^{1/n} \;,$$

n an integer, were considered, but no results were obtained. Considering other forms for h(r) also proved to be fruitless.

The method of quasi-linearization, considered the ratio $\rho(r) = n(r)/m(r)$. The reason for examining this ratio is that it provides good information about the relative changes in n(r) and m(r) as r decreases from R_o to 0. For example, if $\rho(r)$ increases as r decreases from R_o to 0, then Red (n) is "defeating" Blue (m). That is, n is decreasing less rapidly than m. The advantage of the method of quasi-linearization is that it gives a closed-form solution to the Riccati equation 39. The difficulty, of course, is to find a function u(r)

which maximizes (40), or at least find a $u(r)$ which "comes close" to attaining the maximum. One approach is to find a sequence of approximations to the expression given by (40). Theorem 17 gives such a sequence and one which is monotone and uniformly convergent. However, the approximations become rather involved and, hence, are not of too much use in practice. Another way to find a maximizing $u(r)$ is to use a variational calculus approach (see Gelfand and Fomin, 1963). However, this approach was not successful, chiefly because of the "there exists" nature of the theorems in this approach.

4.6 Research Directions

By far the most promising approach to the homogeneous Lanchester problem utilizes the method of successive approximations. Thus, it is natural to expect that further research would involve this technique. One of the first things to consider is how to improve the error bound $E_j(r)$ in the case where $\alpha(r)$ and $\beta(r)$ are both linear. This error bound is valid for very general $\alpha(r)$ and $\beta(r)$, and its value in the linear case in no way uses the linearity of the functions $\alpha(r)$ and $\beta(r)$. Thus, it is natural to expect that a better error bound exists. Notice that the series representations (15) for the solution $(n(r), m(r))$ are such that the signs of consecutive terms alternate. If a condition can be found which guarantees that after the k^{th} term of the series representation, the "tail" of

$n(r)$ becomes an alternating series (i.e., consecutive terms decrease in magnitude as well as have alternating signs), then the error in the partial sum $n_k(r)$ is no more than the absolute value of the last term (i.e., the $(k + 2)$ term) of n_{k+1}. Probably this is the best error bound that can be hoped for, and it is definitely worthwhile attempting to find conditions which force the "tail" to become alternating. The difficulty is that for certain interesting cases of $\alpha(r)$ and $\beta(r)$, the tail may not become an alternating series. In this case other error bounds, such as $E_j(r)$, will have to be used.

Another research direction would be to find $\alpha(r)$ and $\beta(r)$ such that the series solution (15) turned out to be the series representation of a known function. There is no guarantee that there are "interesting" $\alpha(r)$ and $\beta(r)$ for which this is the case, but for the more realistic $\alpha(r)$ and $\beta(r)$ this should at least be considered. This was attempted in the case where $\alpha(r)$ and $\beta(r)$ were both linear using the approximations $n_1(r)$, $n_2(r)$, ..., $n_6(r)$ found in Appendix C, 4, 1. The approach was to see if partial sums of known special functions (see Rainville, 1960) corresponded to the approximations $n_i(r)$. This initial investigation was not fruitful, but it probably would be worthwhile to pursue the approach somewhat further.

Another area of research that should be of great interest and value would be to study the solutions using the approximations $(n_k(r), m_k(r))$ in the case v is a positive function on $[0,R_o]$.

From the form of the series solution (15) together with its derivation it is seen that it is not at all necessary for v to be constant. Once this restriction is removed, it is possible to investigate the problem: Given $\alpha(r)$ and $\beta(r)$, as the combat evolves how should v vary so as to maximize the quantity $m_k(r) - n_k(r)$.

Finally the method of successive approximations can be used to study the heterogeneous-force case with variable coefficients. To see the principle involved, let

$$\phi(r) = \begin{pmatrix} \phi_1(r) \\ \cdots \\ \phi_n(r) \end{pmatrix}$$

and let $A(r)$ be a continuous n x n matrix. Then the system of differential equations

$$\frac{d\phi}{dr} = A(r)\,\phi$$

can be examined using the method of successive approximations just as in the 2 x 2 case. If

$$K(r) = \int_r^{R_o} \sum_{1 \le i,j \le n} |a_{ij}(s)|\,ds, \quad 0 \le r \le R_o \quad ,$$

then the solution $\phi(r)$ and the approximations $\phi_j(r)$ are related by

$$\left\| \phi_j(r) - \phi(r) \right\| < E_j(r) \, ,$$

where

$$E_j(r) = \left[\sum_{i=1}^{n} |\phi_i(R_o)|\right]\left[\exp K(r) - \sum_{\ell=0}^{j} \frac{[K(r)]^\ell}{\ell!}\right].$$

Thus, as in the homogeneous case, a method is available to study the variable-coefficient, heterogeneous-force case in detail.

4.7 References

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Appendix C, 4, 1

SUCCESSIVE APPROXIMATIONS FOR LINEAR ATTRITION-RATE FUNCTIONS

Donald Ballou

The approximations $n_1(r), n_2(r), \ldots, n_6(r)$ to the solution $n(r)$ of theorems 10 and 11 in the text are given below explicitly. It can be shown that n_k and n_{k+1} agree through terms of order $(r - R_o)^k$. Also, n_k is a polynomial in $(r - R_o)$, with $(r - R_o)^{2k}$ being the highest order term. Recall that $\tilde{R}_\alpha = R_\alpha - R_o$ and $\tilde{R}_\beta = R_\beta - R_o$. In the applications we set $R_o = R_\alpha$ to simplify the results, such as Conditions A, B, C. If further approximations are desired, $n_7(r)$ is obtained using equation 10 and the function $m_6(r)$. As mentioned in the text, $m_k(r)$ is obtained from $n_k(r)$ by replacing M with N, N with M, $\tilde{R}_\alpha$ with $\tilde{R}_\beta$, $\tilde{R}_\beta$ with $\tilde{R}_\alpha$, K_α with K_β, and K_β with K_α.

The approximation $n_1(r)$:

$$n_1(r) = N + \frac{K_\alpha \tilde{R}_\alpha M}{v}(r - R_o) - \frac{K_\alpha M}{2v}(r - R_o)^2 .$$

The approximation $n_2(r)$ $\left[\tilde{R}_\alpha = R_\alpha - R_o;\ \tilde{R}_\beta = R_\beta - R_o\right]$:

$$n_2(r) = N + \frac{K_\alpha \tilde{R}_\alpha M}{v}(r - R_o) + \left[\frac{K_\alpha K_\beta \tilde{R}_\alpha \tilde{R}_\beta N}{2v^2} - \frac{K_\alpha M}{2v}\right](r - R_o)^2$$

continued

$$+ \left[- \frac{K_\alpha K_\beta \tilde{R}_\alpha N}{6v^2} - \frac{K_\alpha K_\beta \tilde{R}_\beta N}{3v^2} \right] (r - R_o)^3$$

$$+ \left[\frac{K_\alpha K_\beta N}{8v^2} \right] (r - R_o)^4 .$$

The approximation $n_3(r)$:

$$n_3(r) = N + \frac{K_\alpha \tilde{R}_\alpha M}{v}(r - R_o) + \left[\frac{K_\alpha K_\beta \tilde{R}_\alpha \tilde{R}_\beta N}{2v^2} - \frac{K_\alpha M}{2v} \right] (r - R_o)^2$$

$$+ \left[\frac{K_\alpha^2 K_\beta \tilde{R}_\alpha^2 \tilde{R}_\beta M}{6v^3} - \frac{K_\alpha K_\beta \tilde{R}_\alpha N}{6v^2} - \frac{K_\alpha K_\beta \tilde{R}_\beta N}{3v^2} \right] (r - R_o)^3$$

$$+ \left[- \frac{K_\alpha^2 K_\beta \tilde{R}_\alpha \tilde{R}_\beta M}{6v^3} - \frac{K_\alpha^2 K_\beta \tilde{R}_\alpha^2 M}{12v^3} + \frac{K_\alpha K_\beta N}{8v^2} \right] (r - R_o)^4$$

$$+ \left[\frac{11 \cdot K_\alpha^2 K_\beta \tilde{R}_\alpha M}{120v^3} + \frac{K_\alpha^2 K_\beta \tilde{R}_\beta M}{30v^3} \right] (r - R_o)^5$$

$$+ \left[- \frac{K_\alpha^2 K_\beta M}{48v^3} \right] (r - R_o)^6 .$$

The approximation $n_4(r)$:

$$n_4(r) = N + \left[\frac{K_\alpha \tilde{R}_\alpha M}{v} \right] (r - R_o) + \left[\frac{K_\alpha K_\beta \tilde{R}_\alpha \tilde{R}_\beta N}{2v^2} - \frac{K_\alpha M}{2v} \right] (r - R_o)$$

$$+ \left[\frac{K_\alpha^2 K_\beta \tilde{R}_\alpha^2 \tilde{R}_\beta M}{6v^3} - \frac{K_\alpha K_\beta \tilde{R}_\alpha N}{6v^2} - \frac{K_\alpha K_\beta \tilde{R}_\beta N}{3v^2} \right] (r - R_o)^3$$

continued

$$
+ \left[\frac{K_\alpha^2 K_\beta^2 \tilde{R}_\alpha^2 \tilde{R}_\beta^2 N}{24v^4} - \frac{K_\alpha^2 K_\beta \tilde{R}_\alpha \tilde{R}_\beta M}{6v^3} \right.
$$

$$
\left. - \frac{K_\alpha^2 K_\beta \tilde{R}_\alpha^2 M}{12v^3} + \frac{K_\alpha K_\beta N}{8v^2} \right] (r - R_o)^4
$$

$$
+ \left[\frac{-K_\alpha^2 K_\beta^2 \tilde{R}_\alpha^2 R_\beta N}{30v^4} - \frac{K_\alpha^2 K_\beta^2 \tilde{R}_\alpha \tilde{R}_\beta^2 N}{20v^4} \right.
$$

$$
\left. + \frac{11 \cdot K_\alpha^2 K_\beta \tilde{R}_\alpha M}{120v^3} + \frac{K_\alpha^2 K_\beta \tilde{R}_\beta M}{30v^3} \right] (r - R_o)^5
$$

$$
+ \left[\frac{31 \cdot K_\alpha^2 K_\beta^2 \tilde{R}_\alpha \tilde{R}_\beta N}{16 \cdot 45v^4} + \frac{K_\alpha^2 K_\beta^2 \tilde{R}_\alpha^2 N}{180v^4} + \frac{K_\alpha^2 K_\beta^2 \tilde{R}_\beta^2 N}{72v^4} \right.
$$

$$
\left. - \frac{K_\alpha^2 K_\beta M}{48v^3} \right] (r - R_o)^6
$$

$$
+ \left[\frac{-13 \cdot K_\alpha^2 K_\beta^2 \tilde{R}_\alpha N}{35 \cdot 48v^4} - \frac{11 \cdot K_\alpha^2 K_\beta^2 \tilde{R}_\beta N}{7 \cdot 120v^4} \right] (r - R_o)^7
$$

$$
+ \left[\frac{K_\alpha^2 K_\beta^2 N}{8 \cdot 48v^4} \right] (r - R_o)^8 .
$$

The approximation $n_5(r)$:

$$n_5(r) = N + \frac{K_\alpha \tilde{R}_\alpha M}{v}(r - R_o) + \left[\frac{K_\alpha K_\beta \tilde{R}_\alpha \tilde{R}_\beta N}{2v^2} - \frac{K_\alpha M}{2v}\right](r - R_o)^2$$

$$+ \left[\frac{K_\alpha^2 K_\beta \tilde{R}_\alpha^2 \tilde{R}_\beta M}{6v^3} - \frac{K_\alpha K_\beta \tilde{R}_\alpha N}{6v^2} - \frac{K_\alpha K_\beta \tilde{R}_\beta N}{3v^2}\right](r - R_o)^3$$

$$+ \left[\frac{K_\alpha^2 K_\beta^2 \tilde{R}_\alpha^2 \tilde{R}_\beta^2 N}{24v^4} - \frac{K_\alpha^2 K_\beta \tilde{R}_\alpha \tilde{R}_\beta M}{6v^3} - \frac{K_\alpha^2 K_\beta \tilde{R}_\alpha^2 M}{12v^3} + \frac{K_\alpha K_\beta N}{8v^2}\right](r - R_o)^4$$

$$+ \left[\frac{K_\alpha^3 K_\beta^2 \tilde{R}_\alpha^3 \tilde{R}_\beta^2 M}{120v^5} - \frac{K_\alpha^2 K_\beta^2 \tilde{R}_\alpha^2 \tilde{R}_\beta N}{30v^4} - \frac{K_\alpha^2 K_\beta^2 \tilde{R}_\alpha \tilde{R}_\beta^2 N}{20v^4}\right.$$

$$\left. + \frac{11 \cdot K_\alpha^2 K_\beta \tilde{R}_\alpha M}{120v^3} + \frac{K_\alpha^2 K_\beta \tilde{R}_\beta M}{30v^3}\right](r - R_o)^5$$

$$+ \left[\frac{-K_\alpha^3 K_\beta^2 \tilde{R}_\alpha^2 \tilde{R}_\beta^2 M}{80v^5} - \frac{K_\alpha^3 K_\beta^2 \tilde{R}_\alpha^3 \tilde{R}_\beta M}{120v^5} + \frac{31 \cdot K_\alpha^2 K_\beta^2 \tilde{R}_\alpha \tilde{R}_\beta N}{20 \cdot 36v^4}\right.$$

$$\left. + \frac{K_\alpha^2 K_\beta^2 \tilde{R}_\alpha^2 N}{180v^4} + \frac{K_\alpha^2 K_\beta^2 \tilde{R}_\beta^2 N}{72v^4} - \frac{K_\alpha^2 K_\beta M}{48v^3}\right](r - R_o)^6$$

$$+ \left[\frac{67 \cdot K_\alpha^3 K_\beta^2 \tilde{R}_\alpha^2 \tilde{R}_\beta M}{2^4 \cdot 3^2 \cdot 5 \cdot 7v^5} + \frac{K_\alpha^3 K_\beta^2 \tilde{R}_\alpha \tilde{R}_\beta^2 M}{2^2 \cdot 3^2 \cdot 5v^5} + \frac{K_\alpha^3 K_\beta^2 \tilde{R}_\alpha^3 M}{2^3 \cdot 3^2 \cdot 7v^5}\right.$$

continued

$$- \frac{13 \cdot K_\alpha^2 K_\beta^2 \tilde{R}_\alpha N}{2^4 \cdot 3 \cdot 5 \cdot 7 v^4} - \frac{11 \cdot K_\alpha^2 K_\beta^2 \tilde{R}_\beta N}{2^3 \cdot 3 \cdot 5 \cdot 7 v^5} \Bigg] (r - R_o)^7$$

$$+ \Bigg[\frac{-2 \cdot K_\alpha^3 K_\beta^2 \tilde{R}_\alpha \tilde{R}_\beta M}{3^2 \cdot 5 \cdot 7 v^5} - \frac{17 \cdot K_\alpha^3 K_\beta^2 \tilde{R}_\alpha^2 M}{2^4 \cdot 5 \cdot 7 \cdot 9 v^5}$$

$$- \frac{K_\alpha^3 K_\beta^2 \tilde{R}_\beta^2 M}{2^5 \cdot 3^2 \cdot 5 v^5} + \frac{K_\alpha^2 K_\beta^2 N}{2^7 \cdot 3 v^4} \Bigg] (r - R_o)^8$$

$$+ \frac{211 \cdot K_\alpha^3 K_\beta^2 \tilde{R}_\alpha M}{2^7 \cdot 3^3 \cdot 5 \cdot 7 v^5} + \frac{13 \cdot K_\alpha^3 K_\beta^2 \tilde{R}_\beta M}{2^4 \cdot 3^3 \cdot 5 \cdot 7 v^5} \Bigg] (r - R_o)^9$$

$$+ \left[\frac{-K_\alpha^3 K_\beta^2 M}{2^8 \cdot 3 \cdot 5 v^5} \right] (r - R_o)^{10}.$$

The approximation $n_6(r)$:

$$n_6(r) = N + \frac{K_\alpha \tilde{R}_\alpha M}{v} (r - R_o) + \left[\frac{K_\alpha K_\beta \tilde{R}_\alpha \tilde{R}_\beta N}{2v^2} - \frac{K_\alpha M}{2v} \right] (r - R_o)^2$$

$$+ \left[\frac{K_\alpha^2 K_\beta \tilde{R}_\alpha^2 \tilde{R}_\beta M}{6v^3} - \frac{K_\alpha K_\beta \tilde{R}_\alpha N}{6v^2} - \frac{K_\alpha K_\beta \tilde{R}_\beta N}{3v^2} \right] (r - R_o)^3$$

continued

$$+ \Bigg[\frac{K_\alpha^2 K_\beta^2 \tilde{R}_\alpha^2 \tilde{R}_\beta^2 N}{24v^4} - \frac{K_\alpha^2 K_\beta \tilde{R}_\alpha \tilde{R}_\beta M}{6v^3} - \frac{K_\alpha^2 K_\beta \tilde{R}_\alpha^2 M}{12v^3} + \frac{K_\alpha K_\beta N}{8v^2}\Bigg] (r - R_o)^4$$

$$+ \Bigg[\frac{K_\alpha^3 K_\beta^2 \tilde{R}_\alpha^3 \tilde{R}_\beta^2 M}{120v^5} - \frac{K_\alpha^2 K_\beta^2 \tilde{R}_\alpha^2 \tilde{R}_\beta N}{30v^4} - \frac{K_\alpha^2 K_\beta^2 \tilde{R}_\alpha \tilde{R}_\beta^2 N}{20v^4}$$

$$+ \frac{11 \cdot K_\alpha^2 K_\beta \tilde{R}_\alpha M}{120v^3} + \frac{K_\alpha^2 K_\beta \tilde{R}_\beta M}{30v^3}\Bigg] (r - R_o)^5$$

$$+ \Bigg[\frac{K_\alpha^3 K_\beta^3 \tilde{R}_\alpha^3 \tilde{R}_\beta^3 N}{2^4 \cdot 3^2 \cdot 5v^6} - \frac{K_\alpha^3 K_\beta^2 \tilde{R}_\alpha^2 \tilde{R}_\beta^2 M}{2^4 \cdot 5v^5} - \frac{K_\alpha^3 K_\beta^2 \tilde{R}_\alpha^3 \tilde{R}_\beta M}{2^3 \cdot 3 \cdot 5v^5}$$

$$+ \frac{31 \cdot K_\alpha^2 K_\beta^2 \tilde{R}_\alpha \tilde{R}_\beta N}{2^4 \cdot 3^2 \cdot 5v^4} + \frac{K_\alpha^2 K_\beta^2 \tilde{R}_\alpha^2 N}{2^2 \cdot 3^2 \cdot 5v^4} + \frac{K_\alpha^2 K_\beta^2 \tilde{R}_\beta^2 N}{72v^4} - \frac{K_\alpha^2 K_\beta M}{48v^3}\Bigg] (r - R_o)^6$$

$$+ \Bigg[\frac{-K_\alpha^3 K_\beta^3 \tilde{R}_\alpha^3 \tilde{R}_\beta^2 N}{2^4 \cdot 5 \cdot 7v^6} - \frac{K_\alpha^3 K_\beta^3 \tilde{R}_\alpha^2 \tilde{R}_\beta^3 N}{2^2 \cdot 3 \cdot 5 \cdot 7v^6} + \frac{67 \cdot K_\alpha^3 K_\beta^2 \tilde{R}_\alpha^2 \tilde{R}_\beta M}{2^4 \cdot 3^2 \cdot 5 \cdot 7v^5} + \frac{K_\alpha^3 K_\beta^2 \tilde{R}_\alpha \tilde{R}_\beta^2 M}{2^2 \cdot 3^2 \cdot 5v^5}$$

$$+ \frac{K_\alpha^3 K_\beta^2 \tilde{R}_\alpha^3 M}{2^3 \cdot 3^2 \cdot 7v^5} - \frac{13 \cdot K_\alpha^2 K_\beta^2 \tilde{R}_\alpha N}{2^4 \cdot 3 \cdot 5 \cdot 7v^4} - \frac{11 \cdot K_\alpha^2 K_\beta^2 \tilde{R}_\beta N}{2^3 \cdot 3 \cdot 5 \cdot 7v^4}\Bigg] (r - R_o)^7$$

$$+ \Bigg[\frac{13 \cdot K_\alpha^3 K_\beta^3 \tilde{R}_\alpha^2 \tilde{R}_\beta^2 N}{2^6 \cdot 3^2 \cdot 7v^6} + \frac{K_\alpha^3 K_\beta^3 \tilde{R}_\alpha^3 \tilde{R}_\beta N}{2^5 \cdot 3^2 \cdot 5v^6} + \frac{13 \cdot K_\alpha^3 K_\beta^3 \tilde{R}_\alpha \tilde{R}_\beta^3 N}{2^5 \cdot 3^2 \cdot 5 \cdot 7v^6} - \frac{2 \cdot K_\alpha^3 K_\beta^2 \tilde{R}_\alpha \tilde{R}_\beta M}{3^2 \cdot 5 \cdot 7v^5}$$

continued

$$+\frac{17\cdot K_\alpha^3 K_\beta^2 \tilde{R}_\alpha^2 M}{2^4\cdot 3^2\cdot 5\cdot 7v^5} - \frac{K_\alpha^3 K_\beta^2 \tilde{R}_\beta^2 M}{2^5\cdot 3^2\cdot 5v^5} + \left.\frac{K_\alpha^2 K_\beta^2 N}{2^7\cdot 3v^4}\right] (r - R_o)^8$$

$$+\left[\frac{-K_\alpha^3 K_\beta^3 \tilde{R}_\alpha^2 \tilde{R}_\beta N}{2^2\cdot 3^3\cdot 7v^6} - \frac{K_\alpha^3 K_\beta^3 \tilde{R}_\alpha \tilde{R}_\beta^2 N}{2^2\cdot 3^2\cdot 5v^6} - \frac{K_\alpha^3 K_\beta^3 \tilde{R}_\alpha^3 N}{2^5\cdot 3^4\cdot 5v^6}\right.$$

$$+\frac{211\cdot K_\alpha^3 K_\beta^2 \tilde{R}_\alpha M}{2^7\cdot 3^3\cdot 5\cdot 7v^5} - \frac{K_\alpha^3 K_\beta^3 \tilde{R}_\beta^3 N}{2^3\cdot 3^4\cdot 7v^6} + \left.\frac{13\cdot K_\alpha^3 K_\beta^2 \tilde{R}_\beta M}{2^4\cdot 3^3\cdot 5\cdot 7v^5}\right] (r - R_o)^9$$

$$+\left[\frac{979\cdot K_\alpha^3 K_\beta^3 \tilde{R}_\alpha \tilde{R}_\beta N}{2^8\cdot 3^3\cdot 5^2\cdot 7v^6} + \frac{47\cdot K_\alpha^3 K_\beta^3 \tilde{R}_\alpha^2 N}{2^6\cdot 3^3\cdot 5^2\cdot 7v^6} + \frac{17\cdot K_\alpha^3 K_\beta^3 \tilde{R}_\beta^2 N}{2^5\cdot 3^2\cdot 5^2\cdot 7v^6}\right.$$

$$-\left.\frac{K_\alpha^3 K_\beta^2 M}{2^8\cdot 3\cdot 5v^5}\right] (r - R_o)^{10}$$

$$+\left[\frac{-271\cdot K_\alpha^3 K_\beta^3 \tilde{R}_\alpha N}{2^8\cdot 3^3\cdot 5\cdot 7\cdot 11v^6} - \frac{211\cdot K_\alpha^3 K_\beta^3 \tilde{R}_\beta N}{2^7\cdot 3^3\cdot 5\cdot 7\cdot 11v^6}\right] (r - R_o)^{11}$$

$$+\left[\frac{K_\alpha^3 K_\beta^3 N}{2^{10}\cdot 3^2\cdot 5v^6}\right] (r - R_o)^{12} .$$

Appendix C, 4, 2

ACCURACY OF THE SUCCESSIVE APPROXIMATIONS APPROACH

Donald Ballou

In order to obtain some idea of the accuracy of a given approximation in the case $\alpha(r) = K_\alpha(R_\alpha - r)$ and $\beta(r) = K_\beta(R_\beta - r)$, the following sets of charts were developed. Chart 1 gives the bound on the error in the quantity

$$|n(0) - n_k(0)| + |m(0) - m_k(0)|$$

provided N + M = 100. That is, the number P_{ij} in the i^{th} row and j^{th} column of Chart 1 is such that

$$|n(0) - n_i(0)| + |m(0) - m_i(0)| < P_{ij}$$

whenever the parameters are such that

$$K(0) = \frac{R_\alpha}{2v}\,[K_\alpha R_\alpha + K_\alpha(2R_\beta - R_\alpha)] \leq .5 + (j - 1)(.25)$$

and N + M = 100. If N + M = C, then

$$|n(0) - n_i(0)| + |m(0) - m_i(0)| < P_{ij}\,\frac{C}{100}\,.$$

Charts 2, 3, and 4 give $K(0)\cdot v$ for different values of the parameters K_α, K_β, R_α, R_β. Thus, if R_α = 2000 meters and R_β = 3000 meters, then for $K_\alpha = 10 \times 10^{-6}$ and $K_\beta = 3 \times 10^{-6}$, $K(0)\cdot v = 32$, which implies that $K(0) \leq 2$ if $v \geq 16$. Hence, from Chart 1

$$|n_2(0) - n(0)| + |m_2(0) - m(0)| < 239 ;$$

$$|n_5(0) - n(0)| + |m_5(0) - m(0)| < 12.2 ;$$

$$|n_8(0) - n(0)| + |m_8(0) - m(0)| < .176$$

provided $v \geq 16$ and the parameters have the values given them above.

Chart 1: $E_j(0) = 100 \left[\exp K(0) - \sum_{\ell=0}^{j} \frac{[K(0)]^{\ell}}{\ell!} \right]$

$E_j(0)$ \ $K(0)$	.50	.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
$E_2(0)$	2.37	8.58	21.8	45.9	85.7	147.	239.	371.	558.	811.	1159.
$E_3(0)$	.289	1.54	5.16	13.4	29.4	58.0	106.	181.	259.	465.	709.
$E_4(0)$	2.84×10^{-2}	.225	.995	3.19	8.33	18.9	38.9	74.0	133.	226.	371.
$E_5(0)$	2.38×10^{-3}	2.77×10^{-2}	.162	.642	2.00	5.26	12.2	26.0	51.2	95.2	.169
$E_6(0)$	2.86×10^{-4}	2.96×10^{-3}	2.28×10^{-2}	.112	.415	1.27	3.35	7.94	17.3	35.1	67.3
$E_7(0)$	1.91×10^{-4}	3.82×10^{-4}	2.96×10^{-3}	1.73×10^{-2}	.0761	.270	6.49	2.15	5.17	11.5	23.9
$E_8(0)$	1.91×10^{-4}	1.91×10^{-4}	4.77×10^{-4}	2.48×10^{-3}	.0126	.0516	.176	.522	1.39	3.38	7.64
$E_9(0)$	1.91×10^{-4}	1.91×10^{-4}	2.86×10^{-4}	4.77×10^{-4}	2.00×10^{-3}	9.25×10^{-3}	.0346	.115	.338	.902	2.22
$E_{10}(0)$	1.91×10^{-4}	1.91×10^{-4}	2.86×10^{-4}	2.86×10^{-4}	4.77×10^{-4}	1.91×10^{-3}	6.49×10^{-3}	.0233	.0754	.220	.591

$$K(0)\ v = \frac{R_\alpha}{2}[K_\alpha R_c + K_\beta(2R_\beta - R_\alpha)].$$

Chart 2: Values of $K(0)\cdot v$ if R_α = 2000 meters, R_β = 3000 meters.

K_α \ K_β	2×10^{-6}	3×10^{-6}	4×10^{-6}	5×10^{-6}
6×10^{-6}	20	24	28	32
8×10^{-6}	24	28	32	36
10×10^{-6}	28	32	36	40
12×10^{-6}	32	36	40	44

Chart 3: Values of $K(0)\cdot v$ if R_α = 2500 meters, R_β = 4000 meters.

K_α \ K_β	2×10^{-6}	3×10^{-6}	4×10^{-6}	5×10^{-6}
6×10^{-6}	32.50	39.38	46.25	53.13
8×10^{-6}	38.75	45.63	52.50	59.38
10×10^{-6}	45.00	51.88	58.75	65.63
12×10^{-6}	51.25	58.13	65.00	71.88

Chart 4: Values of K(0)·v if R_α = 1000 meters,
R_β = 1500 meters.

K_α \ K_β	2×10^{-6}	3×10^{-6}	4×10^{-6}	5×10^{-6}
6×10^{-6}	5	6	7	8
8×10^{-6}	6	7	8	9
10×10^{-6}	7	8	9	10
12×10^{-6}	8	9	10	11

Chapter 5

VARIABLE ATTRITION RATES, ANALOG COMPUTER RESULTS

Vernon Larrowe and Raymond Crabtree

The previous chapter indicated the difficulties encountered in attempting to get closed-form, analytical solutions to the coupled differential equations

$$\frac{dn}{dt} = -K_\alpha(R_\alpha - r)m \tag{1}$$

$$\frac{dm}{dt} = -K_\beta(R_\beta - r)n \ , \tag{2}$$

where $dr/dt = v$ and

m = the number of surviving Blue units,

n = the number of surviving Red units,

r = the distance between the Red and Blue forces,

t = the elapsed time since the beginning of battle (t_o),

v = the speed (assumed constant) at which the forces reduce the distance, r, between them,

R_α,[R_β] = the range at which the Blue [Red] forces' weapons first achieve a nonzero attrition rate,

K_α,[K_β] = the constant rate of change of the Blue [Red] weapons attrition rate.

Equations 1 and 2 were programmed for solution on an analog computer to develop some understanding of the important parameters and the underlying dynamics of this description of a battle.

The results of varying parameters of the model such as

K_β, R_α, R_β, R_o (open-fire range) and the initial numbers of forces, M and N, are presented in this chapter. In all cases, except where noted or explicitly varied, R_o is set equal to the larger of R_α, R_β. Since a fairly large number of curves were obtained, it was useful to arrange them in a logical order to explore the behavior of the solutions. Accordingly, each curve is given a four-digit "figure" number, with the digits separated by periods. The significance of each of the digits is described below.

The first digit indicates the basic type of data plotted:

First Digit	Type of Data
1	Solution of the equations at $r = 0$
2	Starting conditions required at $r = R_o$ for a specified outcome at $r = 0$.

The second digit of the figure number indicates the ordinate of the curves:

Second Digit	Ordinate
1	m,n or M,N
2	(m - n) or (M - N)
3	m/n or M/N .

The third digit of the figure number denotes the abscissa of the curves.

Third Digit	Abscissa
1	v
2	K_β
3	R_α
4	not used
5	$\frac{K_\alpha}{v}$
6	$\frac{K_\beta}{v}$
7	$\frac{\sqrt{K_\alpha K_\beta}}{v}$
8	$\frac{K_\alpha}{K_\beta}$
9	r .

The fourth digit of the figure number indicates the parameter which changes from one curve to the next on the figure:

Fourth Digit	Parameter
1	v
2	K_β
3	R_α
4	N
5	$\frac{K_\alpha}{v}$
6	$\frac{K_\beta}{v}$
7	$\frac{\sqrt{K_\alpha K_\beta}}{v}$
8	$\frac{K_\alpha}{K_\beta}$.

In addition, some figure numbers have a letter as a suffix. This letter is used to differentiate between figures whose numbers would be identical otherwise, but in which some quantity changes from one figure to the next or the scaling is different. Thus, Figures 1.1.1.4A and 1.1.1.4B are families of curves which differ only in that in the former, the abscissa, v, goes to 80 meters/second, while in the latter, v only goes to 40 meters/second.

As an example of the figure number coding, consider Figure 1.3.2.1. The meanings of the four digits, taken in order, are

Digit	Meaning
1	This is a plot of conditions at $r = 0$
3	The ordinate is $m(0)/n(0)$
2	The abscissa is K_β
1	Each curve in this figure is for a different value of v.

The figures are contained in Section 5.3. A brief discussion of some of the interesting results is given in the following two sections.

5.1 Solutions at Range r = 0

Figures 1.1.1.2 to 1.3.2.1 show the solutions of the equations at $r = 0$ for various conditions. Figures 1.1.-.- show $m(0)$ and $n(0)$, Figures 1.2.-.- show $[m(0) - n(0)] = d_o$ and Figures 1.3.-.-

show $m(0)/n(0) = \rho_o$. Figures having the same ordinates were grouped together to facilitate comparison.

Figure 1.1.1.2 shows curves of surviving forces [m(0) and n(0)] at r = 0 as a function of closing speed, v. For this situation, $R_\alpha < R_\beta$, but $K_\alpha > K_\beta$ so that the lines $\alpha(r)$ and $\beta(r)$ cross at some value of r between R_α and 0. For Figure 1.1.1.2, both M and N (the initial values of m and n) are 100. The solid curves are for m and the dashed curves are for n. Each curve is labeled according to the value of K_β, which was used in obtaining that curve. Note that each dashed curve (n) has a minimum value at some v. This v, of course, represents the closing speed which gives the fewest surviviors of the Red force. The intersection of a dashed line for a given K_β with the corresponding solid line occurs at a value of v, which results in a "parity" condition (i.e., the surviving Blue forces are equal in number to the surviving Red forces at r = 0). Some of these intersection points are encircled on Figure 1.1.1.2.

Figure 1.1.1.3 shows m(0) and n(0) versus v with R_α as the parameter which varies from one curve to the next. The curves in this figure for R_α = 2000 meters are identical with those for $K_\beta = 5 \times 10^{-6}$ in Figure 1.1.1.2. Increasing R_α increases the Red losses and decreases Blue losses. The curves are somewhat similar to those of Figure 1.1.1.2.

Figure 1.1.1.4A is another set of curves of m(0) and n(0) versus v, but here, N, the initial value of n, is the parameter which is varied from one pair of curves to the next. The array of

curves is similar in appearance to Figures 1.1.1.2 and 1.1.1.3.

Figure 1.1.1.4B is an enlarged version of the left half of Figure 1.1.1.4A. The closing-speed range is from 0-40 meters/second instead of from 0-80 meters/second.

Figure 1.2.1.2 shows the force difference, m - n, at r = 0 as a function of closing speed. K_β is the parameter which is varied from curve to curve. All conditions are the same for this set of curves as for Figure 1.1.1.2. The only differences are the reduction of the range of v from 80 meters/second to 40 meters/second, and plotting of d_o as the ordinate instead of m(0) and n(0

Several features of Figure 1.2.1.2 are of interest: The curves for $K_\beta = 2 \times 10^{-6}$ and for $K_\beta = 3 \times 10^{-6}$ cross the line for $d_o = 0$. The value of v at which these crossings occur represent values at which parity occurs [i.e., m(0) = n(0)]. Note that particularly for $K_\beta = 2 \times 10^{-6}$, the slope of this curve where it intercepts the v-axis is infinite, thus indicating that a very slight increase or decrease of v from 5 meters/second can substantially affect the outcome of the engagement. The outcome at v = 5 meters/second for $K_\beta = 2 \times 10^{-6}$ is indeterminate.

Each of the curves in this figure (1.2.1.2) has a cusp or discontinuity below the v axis. Reference to Figure 1.1.1.2 shows that these discontinuities occur at values of v for which m(0) = 0. In other words, for values of v up to the cusp in each curve, m is wiped out completely. For values of v greater

than that for the cusp, there is some surviving m. The cusp above the v-axis on the curve for $K_\beta = 2 \times 10^{-6}$ occurs where the Red side, n, is completely eliminated. This curve is very sensitive to v. For v < 5 meters/second, Blue is eliminated, and for v = 14 meters/second, Red is eliminated. As v increases, each curve approaches m(0) - n(0) = 0. At infinite speed ($V \rightarrow \infty$), neither side would suffer any losses and the outcome would be m(0) - n(0) = 0.

Figure 1.2.1.3 is a set of curves for m(0) - n(0) versus v, with R_α as the parameter which varies from curve to curve. Here, as in the previous figure, the value of v at which each curve crosses the v-axis represents a parity condition and again some of the curves, particularly those for the higher R_α's, show very high slopes where they cross the v-axis, thus indicating great sensitivity to v at these points. These curves also have cusps or discontinuities. Cusps below the v-axis indicate conditions where m(0) = 0, while those above the v-axis occur for conditions where n(0) = 0.

Figure 1.2.1.4 is a set of curves with abscissae and ordinates the same as for the two preceding figures, but with N, the initial value of n, as the parameter. The v-axis crossings and cusps for these curves have the same significance as these features have in Figures 1.2.1.2 and 1.2.1.3. Note that the curves appear quite similar to those of Figure 1.2.1.3. This

indicates that decreasing N has an effect very similar to that of increasing R_α.

Figure 1.2.2.1 is another presentation of the information shown in Figure 1.2 1.2. The abscissa and parameter have been interchanged, so that now, v is the parameter and K_β is the abscissa. The curves of Figure 1.2.2.1 show an almost linear relationship between m(0) - n(0) and K_β, except for the curve for v = 10, which has discontinuities. In Figure 1.2.1.2, the vertical line for v = 10 passes through the region of discontinuities for the curves of constant K_β, so it is to be expected that the transformation of this line to the n(0) - m(0), K_β coordinate system of Figure 1.2.2.1 would show discontinuities.

Figures 1.3.1.2, 1.3.1.3 and 1.3.1.4A and B are curves showing the final force ratio, $\rho_o = \frac{m(0)}{n(0)}$, as a function of v. The parameter which is varied from curve to curve for Figure 1.3.1.2 is K_β, that for Figure 1.3.1.3 is R_α, and that for Figures 1.3.1.4A and B is N. Figure 1.3.1.4B is similar to Figure 1.3.1.4A except that the v-axis has been extended to 80 meters/second.

For these figures which show ρ_o as the ordinate, the point where a curve crosses the line $\rho_o = 1$ represents the parity condition. Any point above this line indicates a superiority of forces for the blue side, m, and any point below this line represents superiority of forces for red. These families of curves are very similar in appearance, regardless of whether K_β, R_α, or N is the parameter.

Figure 1.3.2.1 shows ρ_o as a function of K_β with v as a parameter. It is similar to Figure 1.2.2.1, with ρ_o as the ordinate instead of m(0) - n(0). The curve for v = 10, which is discontinuous in Figure 1.2.2.1, is not reproduced in Figure 1.3.2.1; however, the curve for v = 20 in Figure 1.2.2.1 is almost linear in this figure, and shows definite curvature in Figure 1.3.2.1.

5.2 Initial Conditions to Achieve a Specific Outcome

The 2.-.-.- series of curves show various sets of initial conditions and parameters required to give m(0) = n(0) = 10 at r = 0. This is a specific parity condition, where m(0) - n(0) = 0 and m(0)/n(0) = 1.

Data for these curves were obtained by setting conditions on the integrators of the analog computer circuit for the desired outcome of the engagement [m(0) = n(0) = 10] and operating the circuit backwards (in negative time) until R_o = 3000 meters.

Figure 2.1.1.3 shows the required values of M and N, as functions of v, which will lead to an outcome of m(0) = n(0) = 10. The parameter, R_α, goes from 2000 to 3000 meters. The dashed curve labeled "2" is N versus v for an R_α of 2000 meters. The dashed curve labeled "3" is N versus v for an R_α of 3000 meters. The dashed curves between these two are for intermediate values of R_α at intervals of 200 meters. The curve for 2400 meters was omitted, since it would have fallen on the M curves.

The solid curves in this figure are for M versus v and there is a curve for each R_α from 2000 meters to 3000 meters at 200-meter intervals. They occur in the same order as the N curves. The one giving the lowest M for a particular v is for R_α = 2000 meters, while the one giving the highest M for this v is for R_α = 3000 meters. In this figure, it appears that for some R_α between 2200 and 2400 meters the N versus v curve could almost coincide with the corresponding M versus v curve. If this were true the implication would be that for this R_α, the values of M and N would always be equal, regardless of v, if the outcome were to be m(0) = n(0) = 10.

Figure 2.1.1.5 is a plot of starting conditions, as a function of v, to give m(0) = n(0) = 10 at the outcome, with K_α and K_β varied so that K_α/v and K_β/v remained constant for the various values of v. This figure shows that under these conditions, M and N are independent of v.

The validity of this conclusion may be shown analytically. In a straightforward manner (1) and (2) can be transformed to

$$\frac{dn}{dr} = \frac{K_\alpha}{v}(R_\alpha - r)m \tag{3}$$

and

$$\frac{dm}{dr} = \frac{K_\beta}{v}(R \ - r)n \ . \tag{4}$$

If v is constant, then K_α/v and K_β/v will also be constant. If v is changed, but K_α and K_β are readjusted to make K_α/v

and K_β/v remain unchanged, equations 3 and 4 are unchanged. The outcome for given values of M and N will then be independent of v, although changing v will change the rate at which the solution is generated in the analog computer circuit.

Figure 2.1.3.1 shows the same data as that plotted in Figure 2.1.1.3, but this time R_α is the abscissa and v is the parameter. The intersection between each solid line (M) and the corresponding dashed line (N) for a given v represents the initial values and value of R_α, for equal numbers of m and n at the beginning of the engagement, as well as equal numbers at the end. These points do not fall along a line of constant R_α as was implied by Figure 2.1.1.3.

Figures 2.1.5.6A through 2.1.5.6I, inclusive, are plots of M and N versus K_α/v, with K_β/v as the parameter. The initial conditions defined by the curves will result in an outcome of $m(0) = n(0) = 10$. For Figures 2.1.5.6A through 2.1.5.6D, R_α is at 1500 meters and R_β is at 3000 meters, but R_o is varied from figure to figure. Values of R_o for these four figures are 3000 meters, 2250 meters, 1500 meters, and 750 meters, respectively. Although these four figures contain considerable information, it is difficult to draw any general conclusions from examining them. It appears that, for smaller values of R_o, the values of M and N needed to produce an outcome of $m(0) = n(0) = 10$ are reduced, as would be expected. One way of interpreting the data in these four figures is to regard each of them as

a picture of the situation at one of four successive values of R_o. Thus, Figure 2.1.5.6A gives the values of M and N needed at R_o = 3000 meters if the outcome at r = 0 is to be m(0) = n(0) = 10. If K_α, K_β and v remain constant, r, the distance between the forces, becomes smaller at a steady rate. When it reaches 2250 meters, the values of m and n at this point may be found by referring to Figure 2.1.5.6B and reading off the M and N for the assumed constant values of K_α, K_β, and v. Figure 2.1.5.6C gives the "picture" when R_o has diminished to 1500 meters, and Figure 2.1.5.6D gives the information when R_o = 750 meters. It is apparent that as R_o approaches 0, the lines of M and N versus K_α/v will become more horizontal and will eventually coincide with the line, M = N = 10.

For Figures 2.1.5.6E through 2.1.5.6I, R_β is at 3000 meters, but R_α has been increased to $\frac{3}{4}R_\beta$ or 2250 meters. This increase in range of the Blue forces' weapon would be expected to raise the initial strength of the Red forces and possibly reduce the initial strength of the Blue force over those for figures 2.1.5.6A to 2.1.5.6D, where R_α was only one half of R_β.

Comparison of Figures 2.1.5.6A and 2.1.5.6E shows that for any given K_α/v and K_β/v, the change of R_α from $\frac{1}{2}R_\beta$ to $\frac{3}{4}R_\beta$ does increase the required N at R_o = 3000 meters, but it also increases the required M. This required increase in M is somewhat unexpected, and should be investigated further.

Figures 2.1.7.8A, 2.1.7.8B, and 2.1.7.8C are plots of M and N versus $\sqrt{K_\alpha K_\beta}/v$ with K_α/K_β as the parameter. Again, these values of M and N are for an outcome of $m(0) = n(0) = 10$. Specification of a value of $\sqrt{K_\alpha K_\beta}/v$ and a value for K_α/K_β is equivalent to specifying K_α/v and K_β/v. Thus, if[1]

$$\frac{\sqrt{K_\alpha K_\beta}}{v} = x \tag{5}$$

and

$$\frac{K_\alpha}{K_\beta} = y \tag{6}$$

then

$$\frac{K_\alpha K_\beta}{v^2} = x^2 \quad , \tag{7}$$

and (6) and (7) may be solved for K_α/v and K_β/v to give

$$\frac{K_\alpha}{v} = x\sqrt{y} \qquad x,y > 0 \tag{8}$$

and

$$\frac{K_\beta}{v} = \frac{x}{\sqrt{y}} \qquad x,y > 0 . \tag{9}$$

Since it was previously shown that M and N remain constant when K_α/v and K_β/v are constant even though v is changed, the

[1]The reader is referred to Section 4.2, where the value of the dimensionless parameter R_Δ, which is a function of x, is discussed.

curves of Figures 2.1.7.8A, 2.1.7.8B, and 2.1.7.8C are valid for all $v > 0$.

An interesting feature of these three figures is the lack of crossings of the "M" lines with the "N" lines. It appears that if $M \geq N$ for any value of $\sqrt{K_\alpha K_\beta}/v$, this relationship holds true for all values of $\sqrt{K_\alpha K_\beta}/v$. This is experimental data only, and the validity should be investigated further, but the noncrossing condition certainly appears to hold for Figures 2.1.7.8A, 2.1.7.8B, and 2.1.7.8C where R_o varies from 3000 to 750 meters.

Figure 2.2.1.3 shows curves of $M - N$, the difference in initial forces, as a function of v, which will give an engagement outcome of $m(0) = n(0) = 10$. The parameter is R_α. This figure was plotted from the same data as that used for Figure 2.1.1.3, the difference being that $M - N$ instead of M and N is the ordinate. This figure shows that there is no value of R_α such that $M = N$ for all values of v, although the curve for R = 2200 meters shows $M \approx N$ for $v > 50$ meters/second.

Figure 2.2.3.1 has the same abscissa and parameter as Figure 2.1.3.1, but the ordinate is $M - N$ instead of M and N. The curves for $v = 50$ and $v = 80$ are also for $M - N$; they were plotted in dashed form to help identify them on each side of the intersections with other curves.

Figures 2.2.5.6A through 2.2.5.6I are for the same abscissa, parameters, and conditions on R_α, R_β, and R_o as Figures 2.1.5.6A though 2.1.5.6I, respectively, but with ordinates of $M - N$

instead of M and N. Again, these are plots of initial conditions which will lead to $m(0) = n(0) = 10$ at the end of the engagement. These curves do not give complete information, as the actual initial values of m and n must be given, rather than M - N, in order to guarantee that the outcome will be $m(0) = n(0) = 10$. They were plotted to give an indication of how M - N behaves. Points of interest are (1) the plots are almost straight lines, and (2) points where M - N curves intersect the line M - N = 0 represent conditions where the Blue and Red forces start with equal numbers and the engagement terminates in a parity condition. Conditions where the curves go below the (M - N = 0)-axis represent conditions where the Red force is larger at the beginning of the engagement.

Figures 2.2.7.8A through 2.2.7.8C are plots of the initial force difference, M - N versus $\sqrt{K_\alpha K_\beta}/v$ with K_α/K_β as the parameter. These curves are plots of the differences of the M and N curves of Figures 2.1.7.8A through 2.1.7.8C, respectively. They have the same general appearance as the M and N curves, themselves.

Figure 2.3.1.3 is a plot of M/N (for an outcome of $m(0) = n(0) > 10$) versus v, with R_α as the parameter. The abscissa and parameter is the same for this figure as for Figure 2.1.3.1; only the ordinate has been changed from M and N to M/N. The region below M/N = 1 represents conditions where the initial strength of the Blue forces is less than that of the Red, and

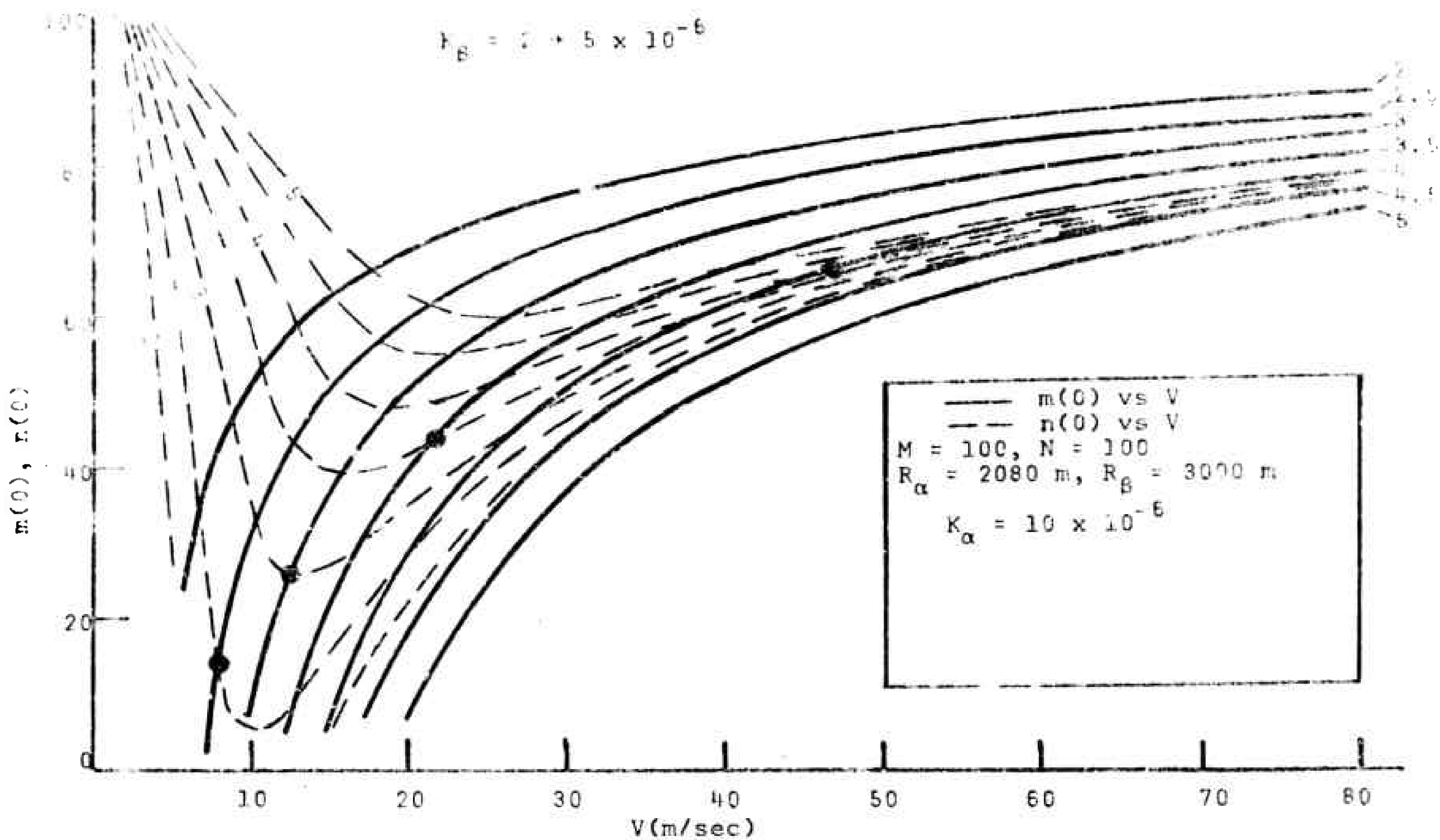
thus represents the condition of Blue defeating more of Red than it loses, since at $r = 0$ the two forces are equal.

Figure 2.3.3.1 shows the same data as that of Figure 2.3.1.3, but with the abscissa and parameter interchanged. This figure is interesting because it shows that the relationship between R_α and M/N for a constant v is almost linear.

Figures 2.3.5.6A through 2.3.5.6I show M/N versus K_α/v, with K_β/v as the parameter. These figures correspond to Figures 2.1.5.6A through 2.1.5.6I, respectively, with M/N as the ordinate instead of M and N.

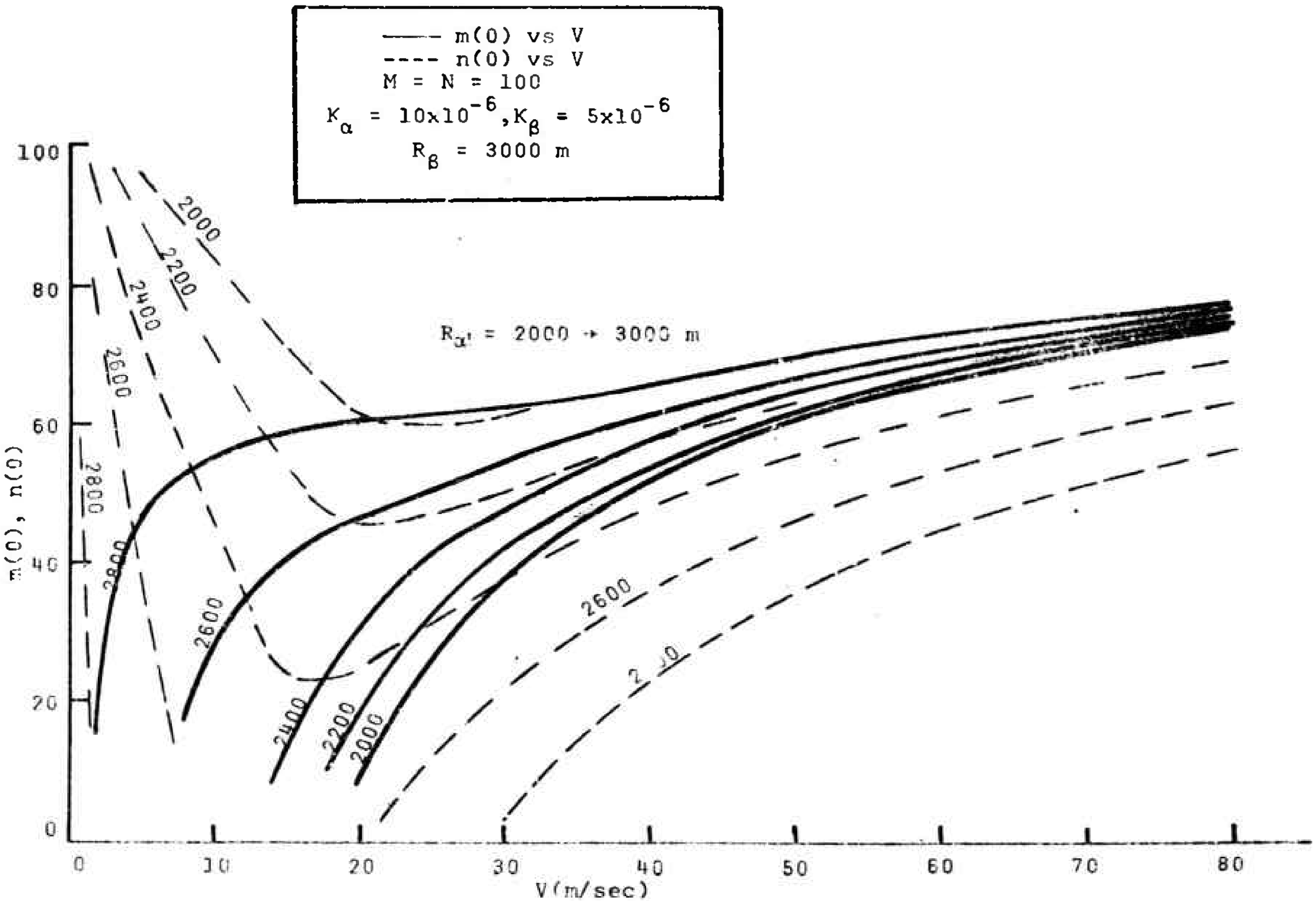
Figures 2.3.7.8A through 2.3.7.8C correspond to Figures 2.1.7.8A through 2.1.7.8C, with M/N as the ordinate instead of M and N. They use $\sqrt{K_\alpha K_\beta}/v$ as the abscissa and K_α/K_β as the parameter.

5.3 Figures Showing Results of Parametric Variations



$K_\beta = 2 \rightarrow 5 \times 10^{-6}$
m(0) vs V
n(0) vs V
M = 100, N = 100
R_α = 2080 m, R_β = 3000 m
$K_\alpha = 10 \times 10^{-6}$
m(0), n(0)
V(m/sec)
0
20
40
10
20
30
40
50
60
70
80
5

Figure 1.1.1.2



—— m(0) vs V
---- n(0) vs V
M = N = 100
$K_\alpha = 10\times10^{-6}, K_\beta = 5\times10^{-6}$
$R_\beta = 3000$ m
$R_{\alpha'} = 2000 \to 3000$ m
2000
2200
2400
2600
2800
m(0), n(0)
V(m/sec)
0
10
20
30
40
50
60
70
80
100

Figure 1.1.1.3

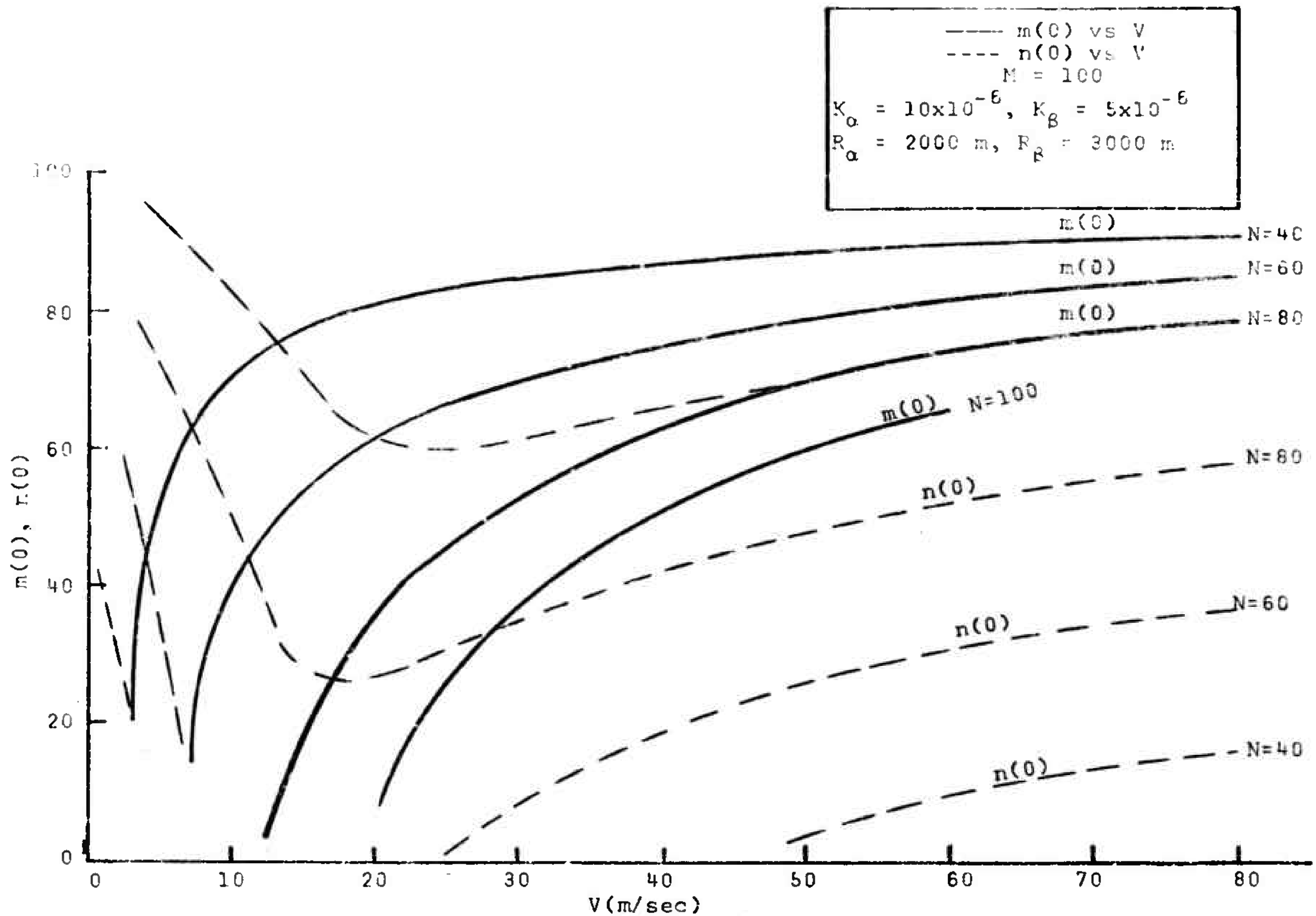


Figure 1.1.1.4A

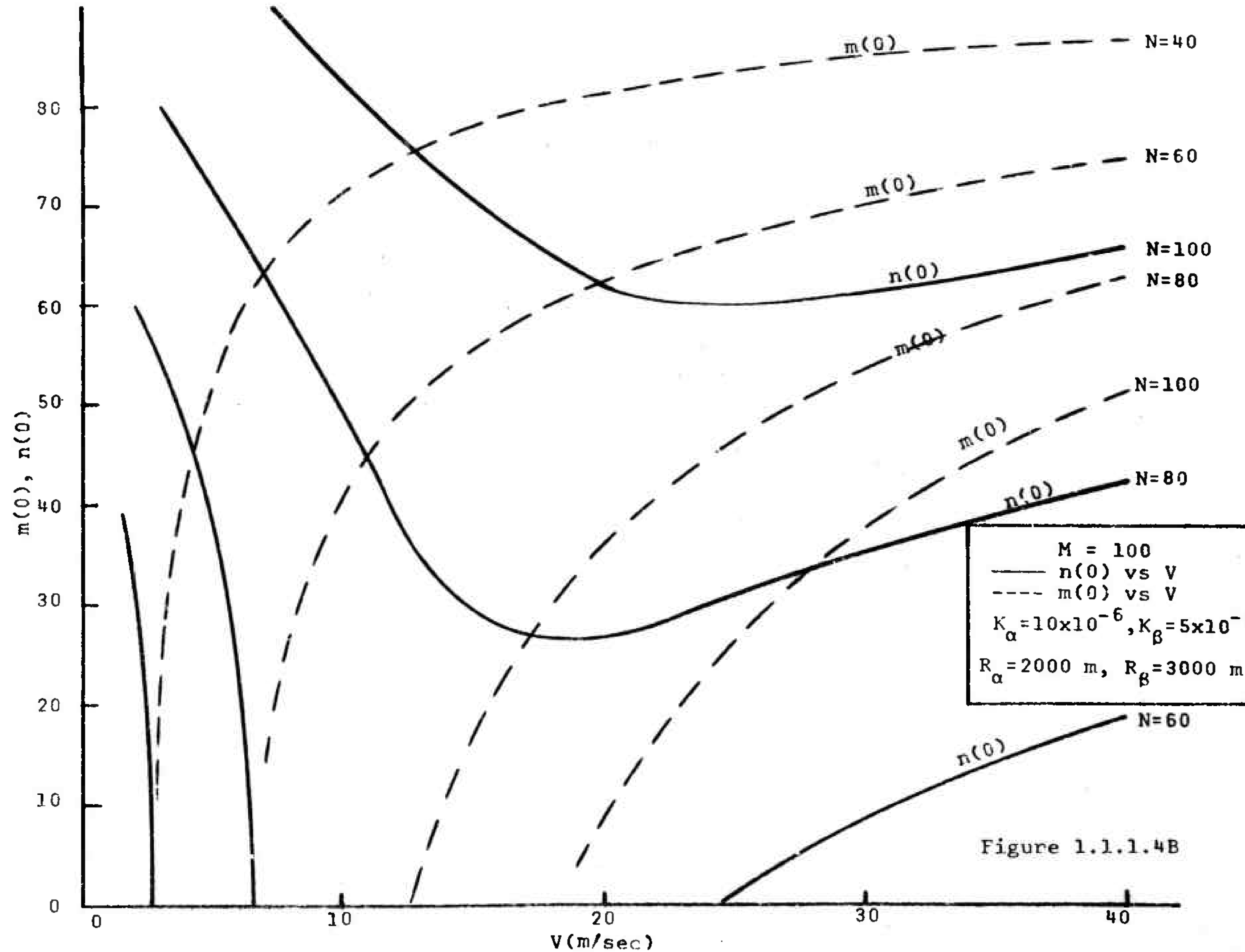


Figure 1.1.1.4B

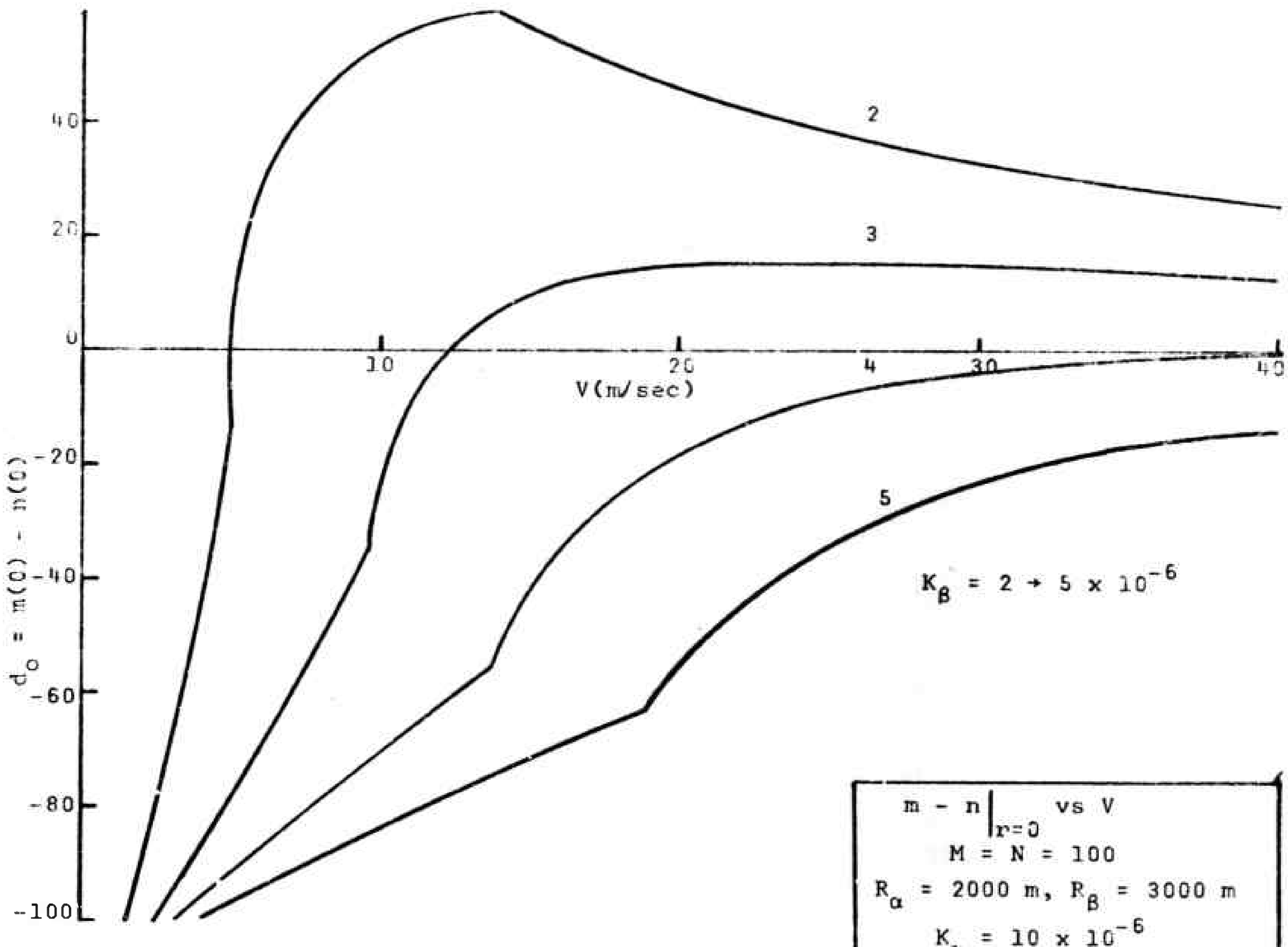

40
20
0
-20
-40
-60
-80
-100
$d_o = m(0) - n(0)$
10
20
30
40
V(m/sec)
2
3
4
5
$K_\beta = 2 \rightarrow 5 \times 10^{-6}$
$m - n\big|_{r=0}$ vs V
M = N = 100
R_α = 2000 m, R_β = 3000 m
$K_\alpha = 10 \times 10^{-6}$

Figure 1.2.1.2

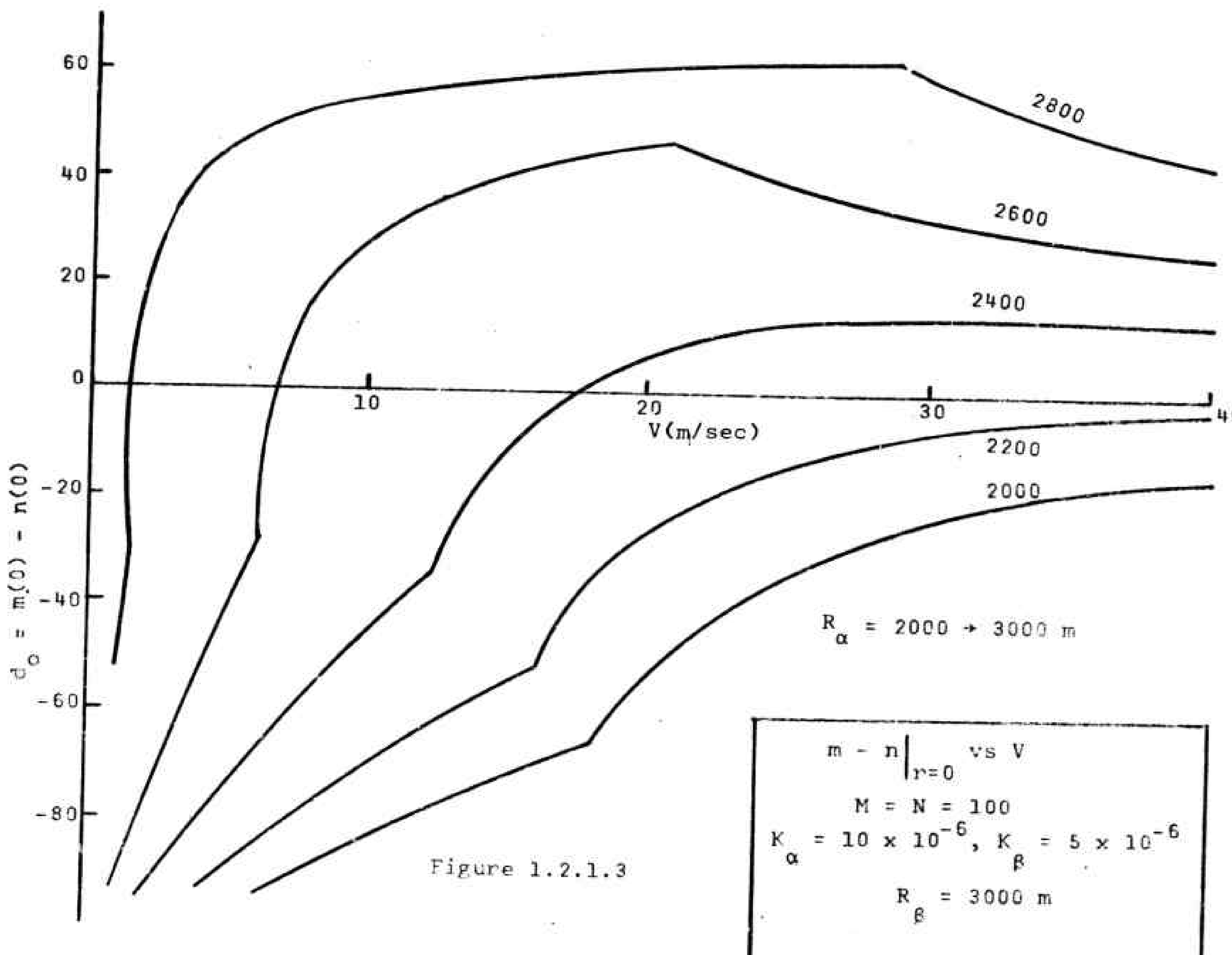

60
40
20
0
-20
-40
-60
-80
$d_o = m(0) - n(0)$
10
20
30
40
V(m/sec)
2800
2600
2400
2200
2000
$R_\alpha = 2000 \rightarrow 3000$ m
$m - n\big|_{r=0}$ vs V
M = N = 100
$K_\alpha = 10 \times 10^{-6}$, $K_\beta = 5 \times 10^{-6}$
$R_\beta = 3000$ m

Figure 1.2.1.3

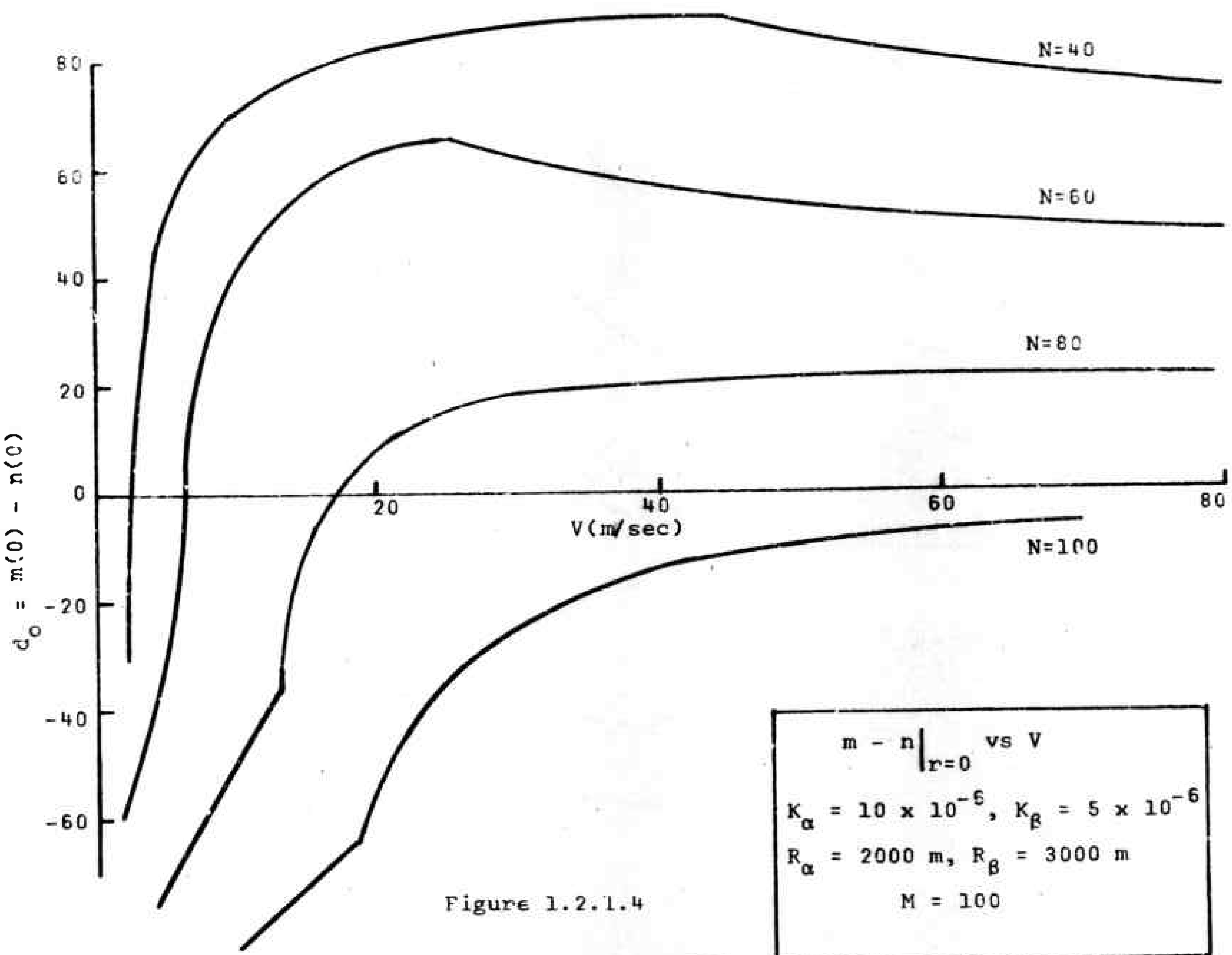

$d_o = m(0) - n(0)$
80
60
40
20
0
-20
-40
-60
20
40
60
80
V(m/sec)
N=40
N=60
N=80
N=100
$m - n\big|_{r=0}$ vs V
$K_\alpha = 10 \times 10^{-6}$, $K_\beta = 5 \times 10^{-6}$
$R_\alpha = 2000$ m, $R_\beta = 3000$ m
M = 100

Figure 1.2.1.4

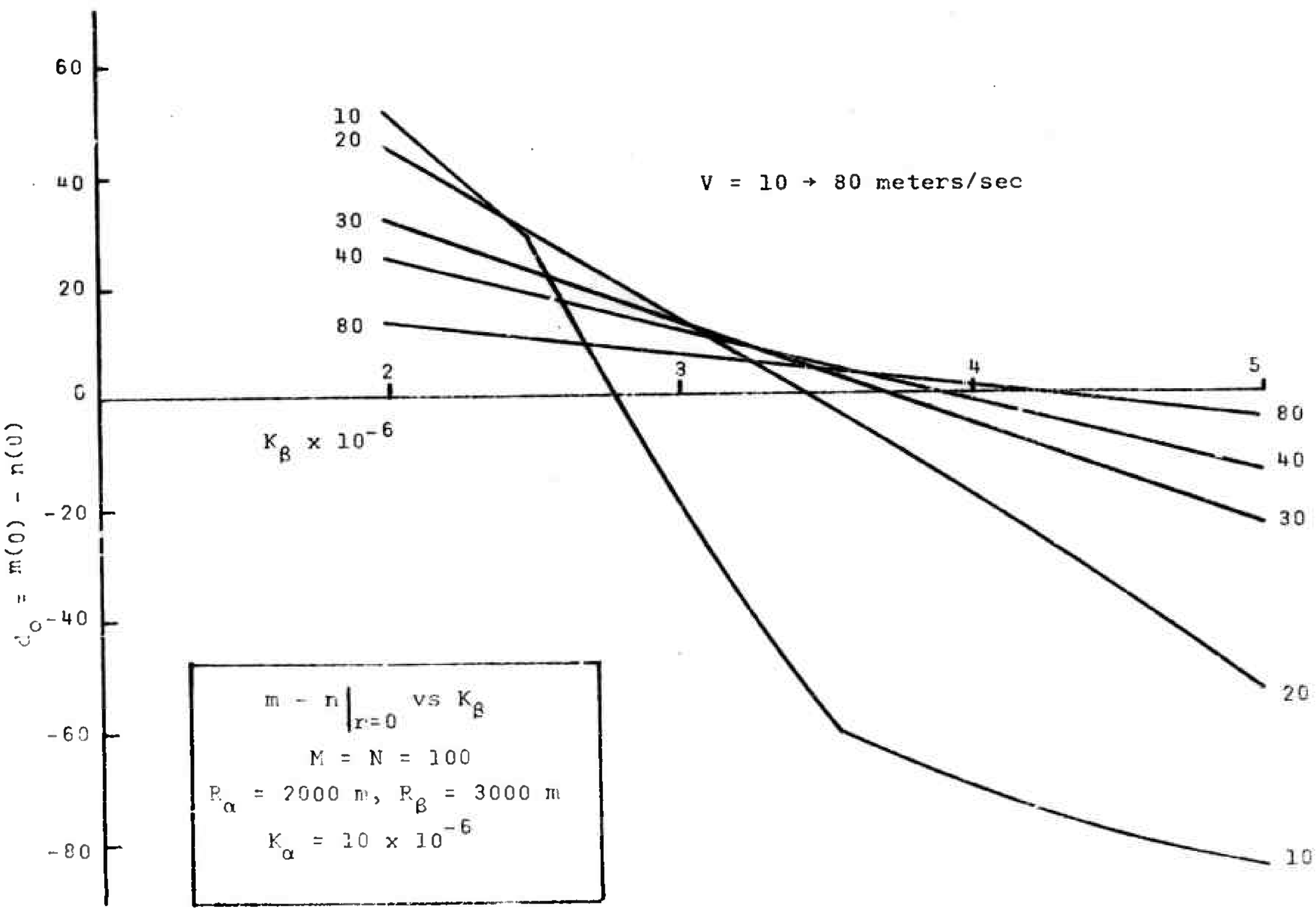

V = 10 → 80 meters/sec
$c_o = m(0) - n(0)$
60
40
20
0
-20
-40
-60
-80
2
3
4
5
$K_\beta \times 10^{-6}$
10
20
30
40
80
$m - n\big|_{r=0}$ vs K_β
M = N = 100
R_α = 2000 m, R_β = 3000 m
$K_\alpha = 10 \times 10^{-6}$

Figure 1.2.2.1

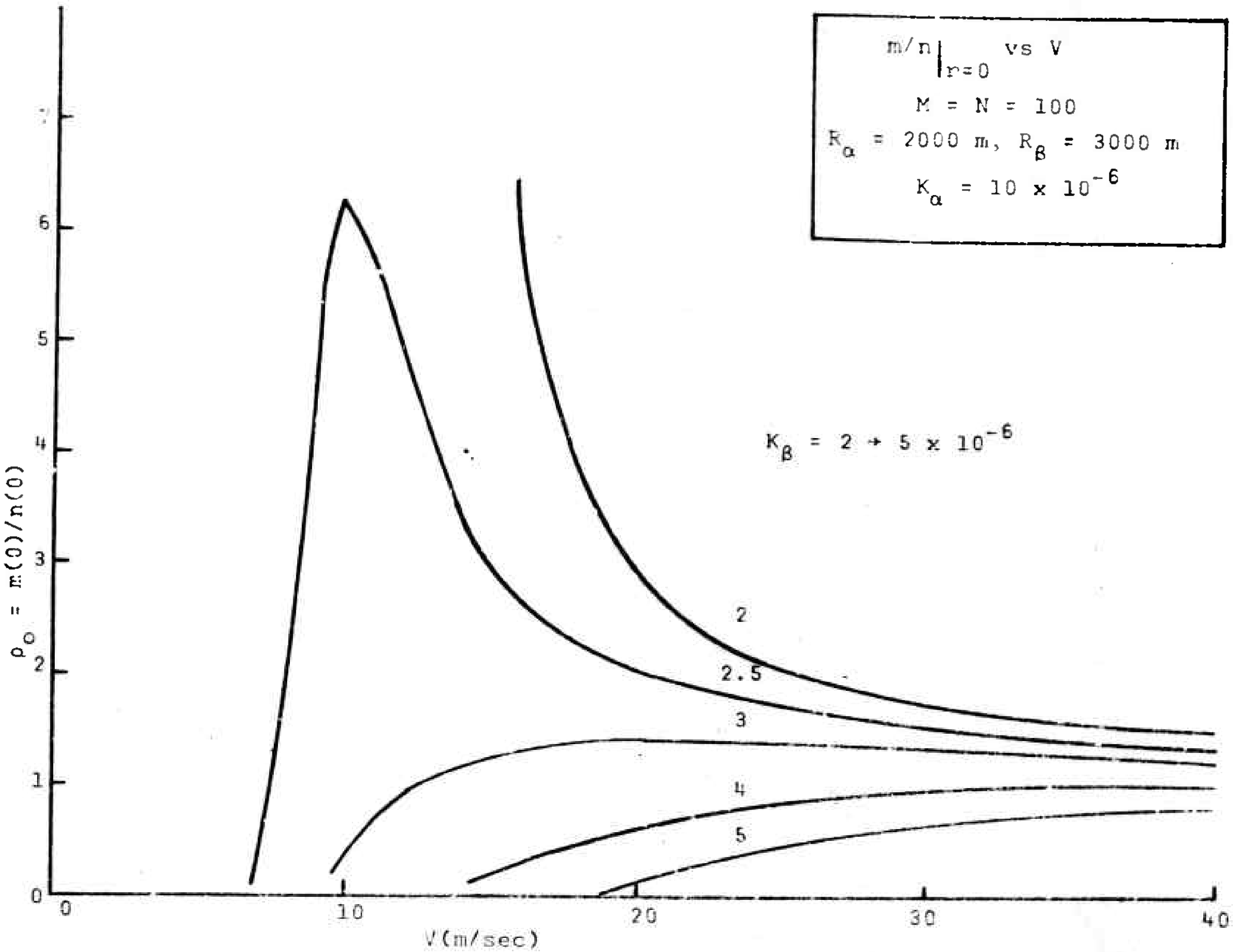

m/n|r=0 vs V
M = N = 100
Rα = 2000 m, Rβ = 3000 m
Kα = 10 x 10^-6
Kβ = 2 → 5 x 10^-6
2
2.5
3
4
5
ρo = m(0)/n(0)
0
1
2
3
4
5
6
0
10
20
30
40
V(m/sec)

Figure .3

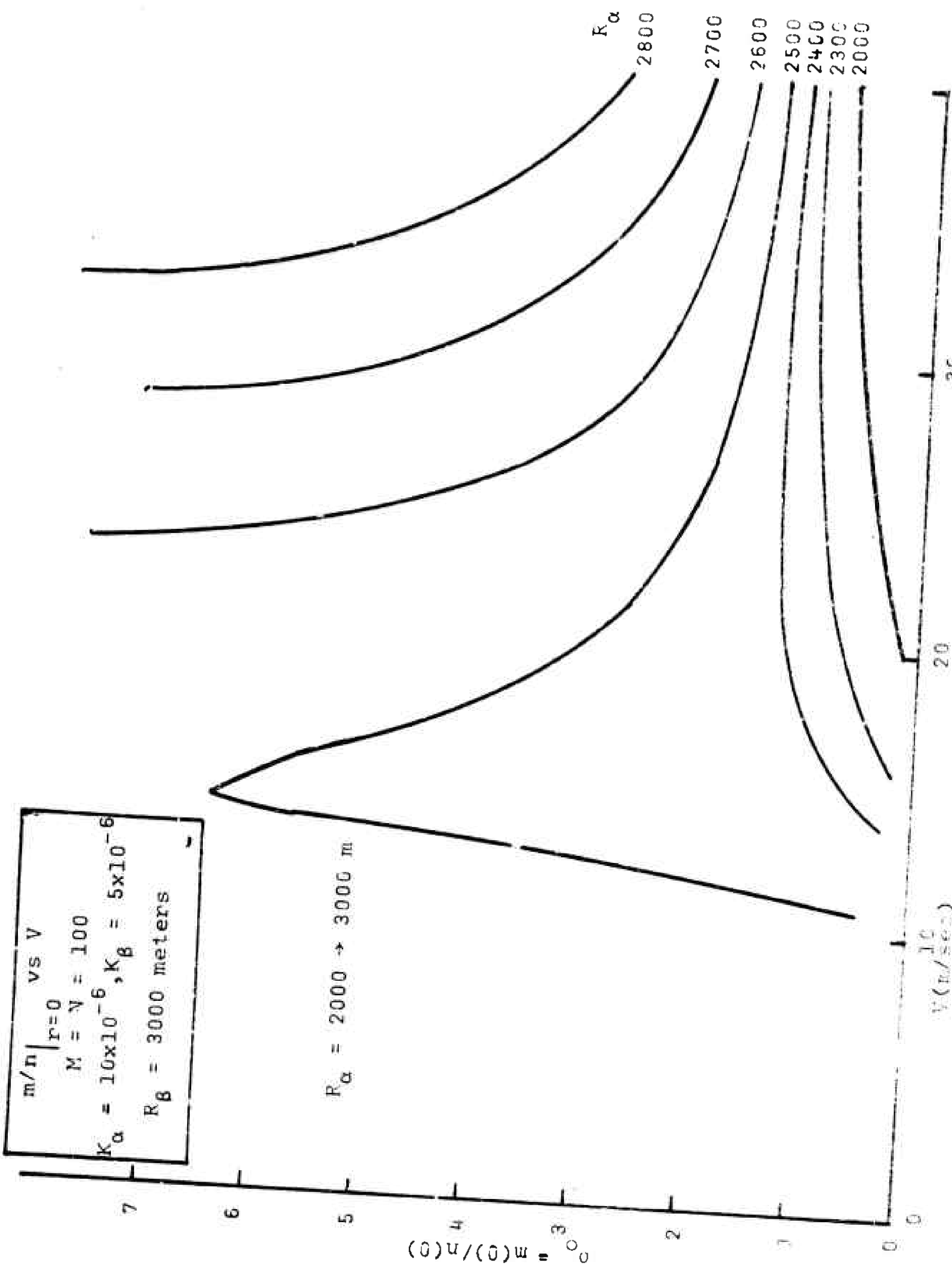

m/n|r=0 vs V
M = N = 100
Kα = 10x10^-6, Kβ = 5x10^-6
Rβ = 3000 meters
Rα = 2000 → 3000 m
Rα
2800
2700
2600
2500
2400
2300
2000
0
10
20
30
40
V(m/sec)
0
1
2
3
4
5
6
7
co = m(0)/n(0)

Figure 1.3.1.3

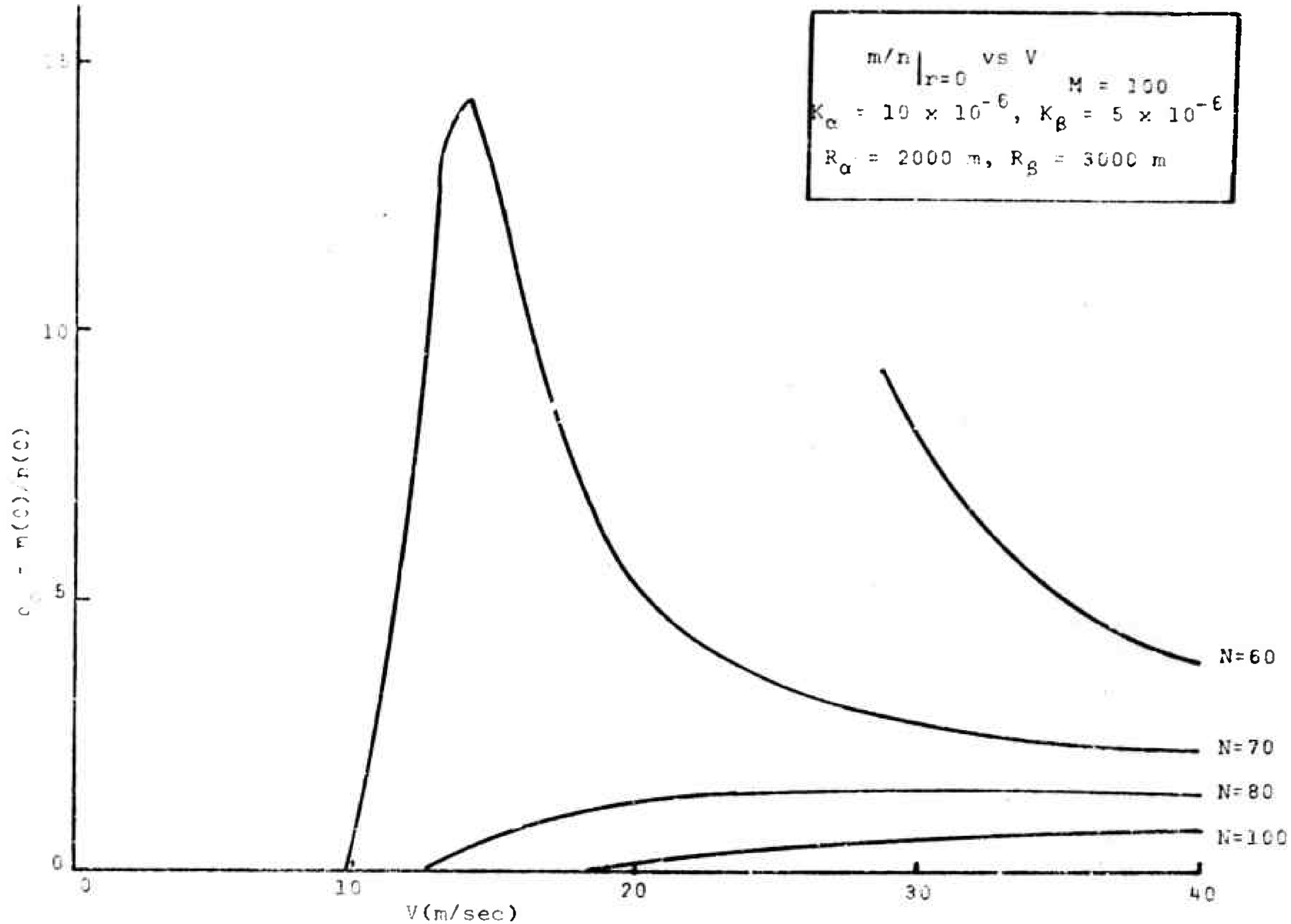


Figure 1.3.1.4A

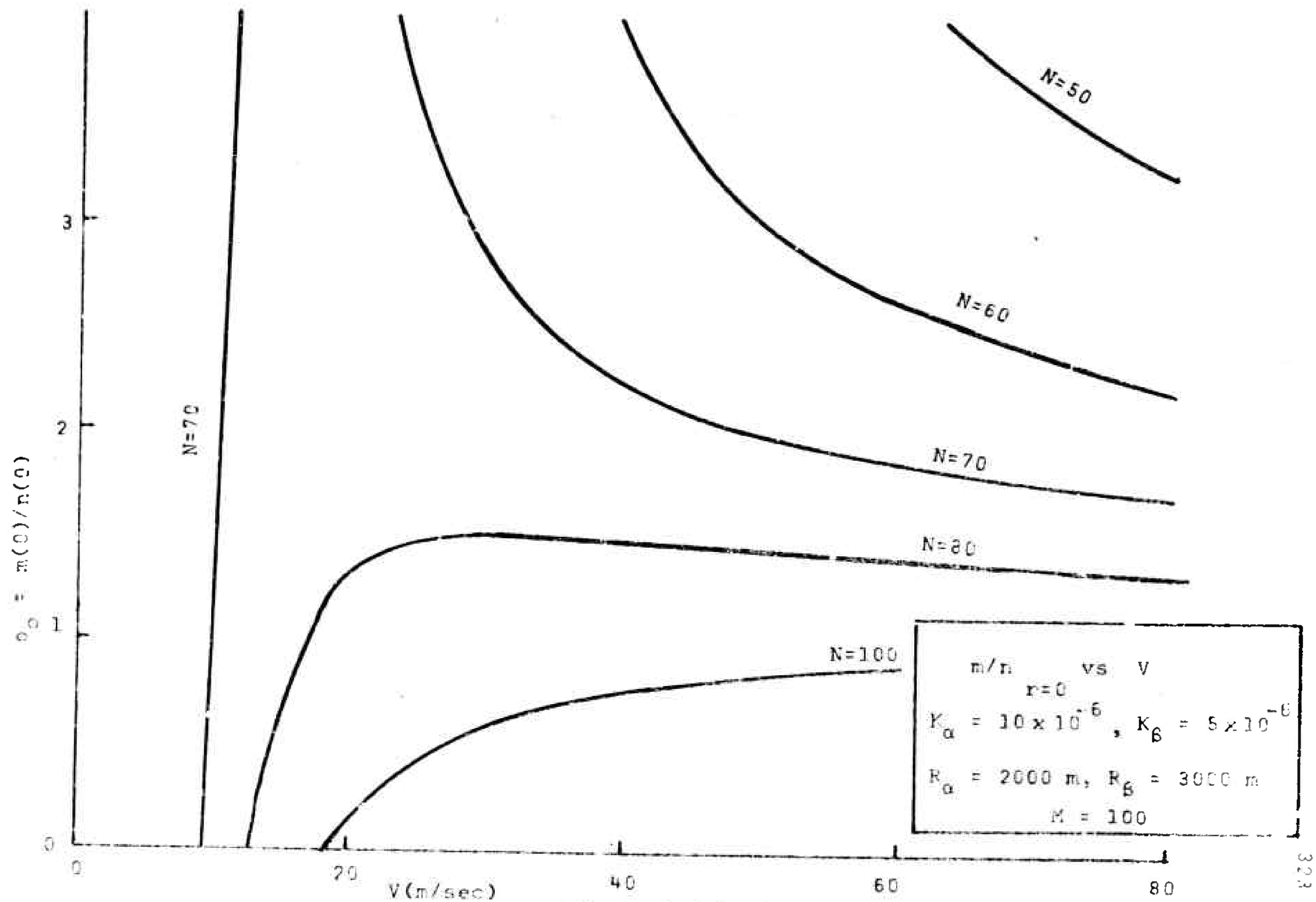
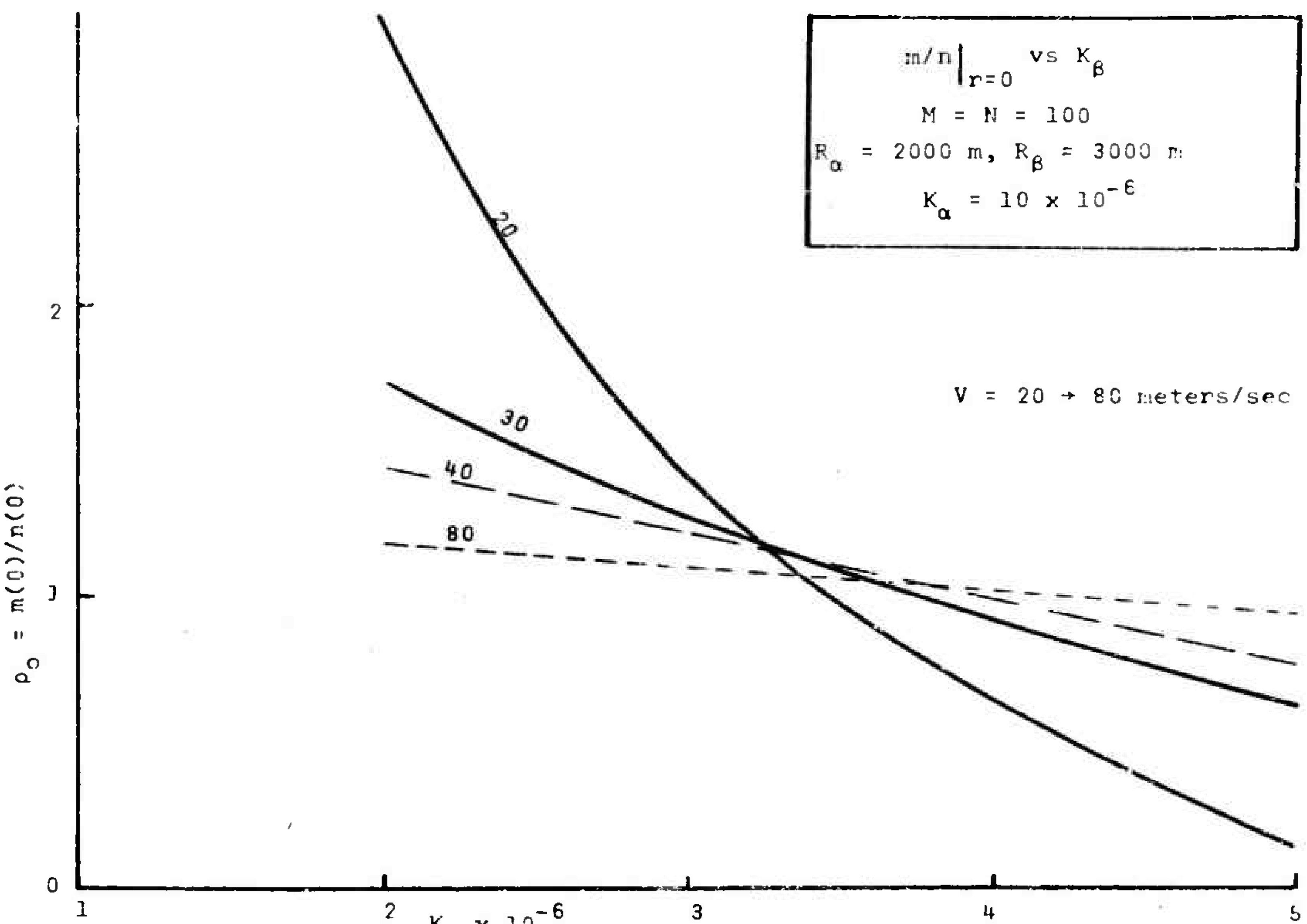


Figure 1.3.1.4B



$m/n|_{r=0}$ vs K_β
$M = N = 100$
$R_\alpha = 2000$ m, $R_\beta = 3000$ m
$K_\alpha = 10 \times 10^{-6}$
$V = 20 \rightarrow 80$ meters/sec
20
30
40
80
$\rho_0 = m(0)/n(0)$
0
1
2
1
2
3
4
5
$K_\beta \times 10^{-6}$

Figure 1.3.2.1

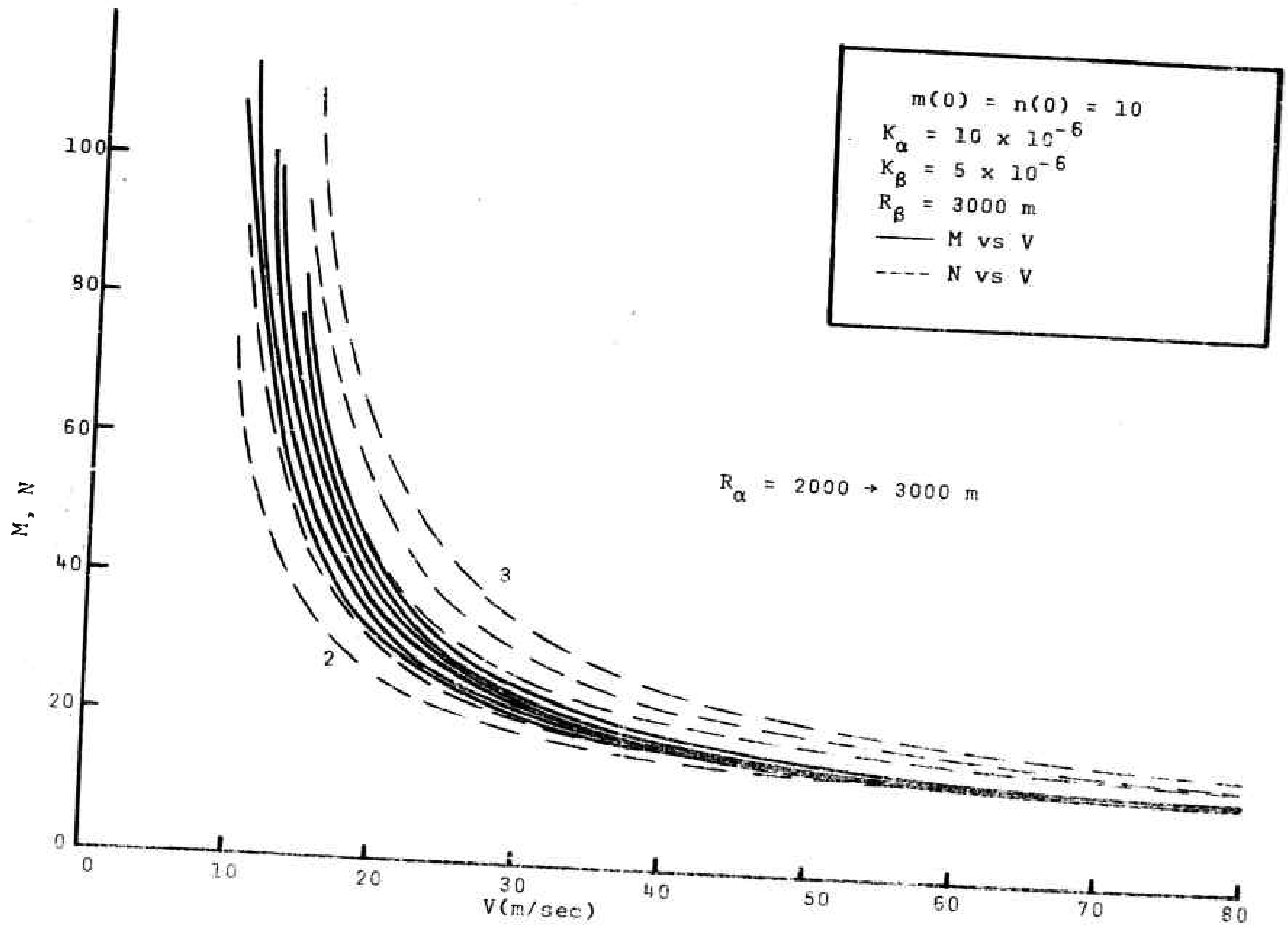
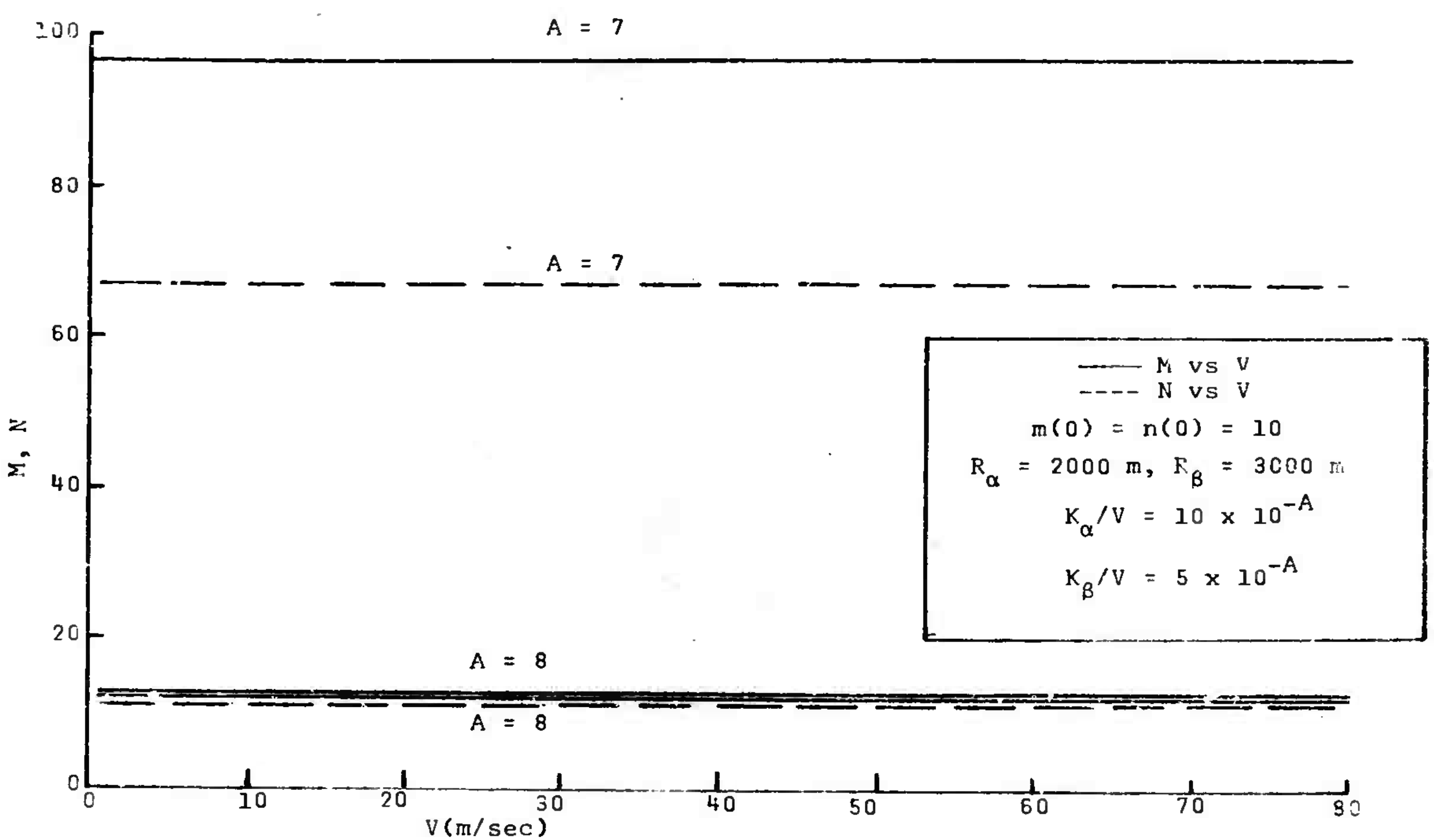
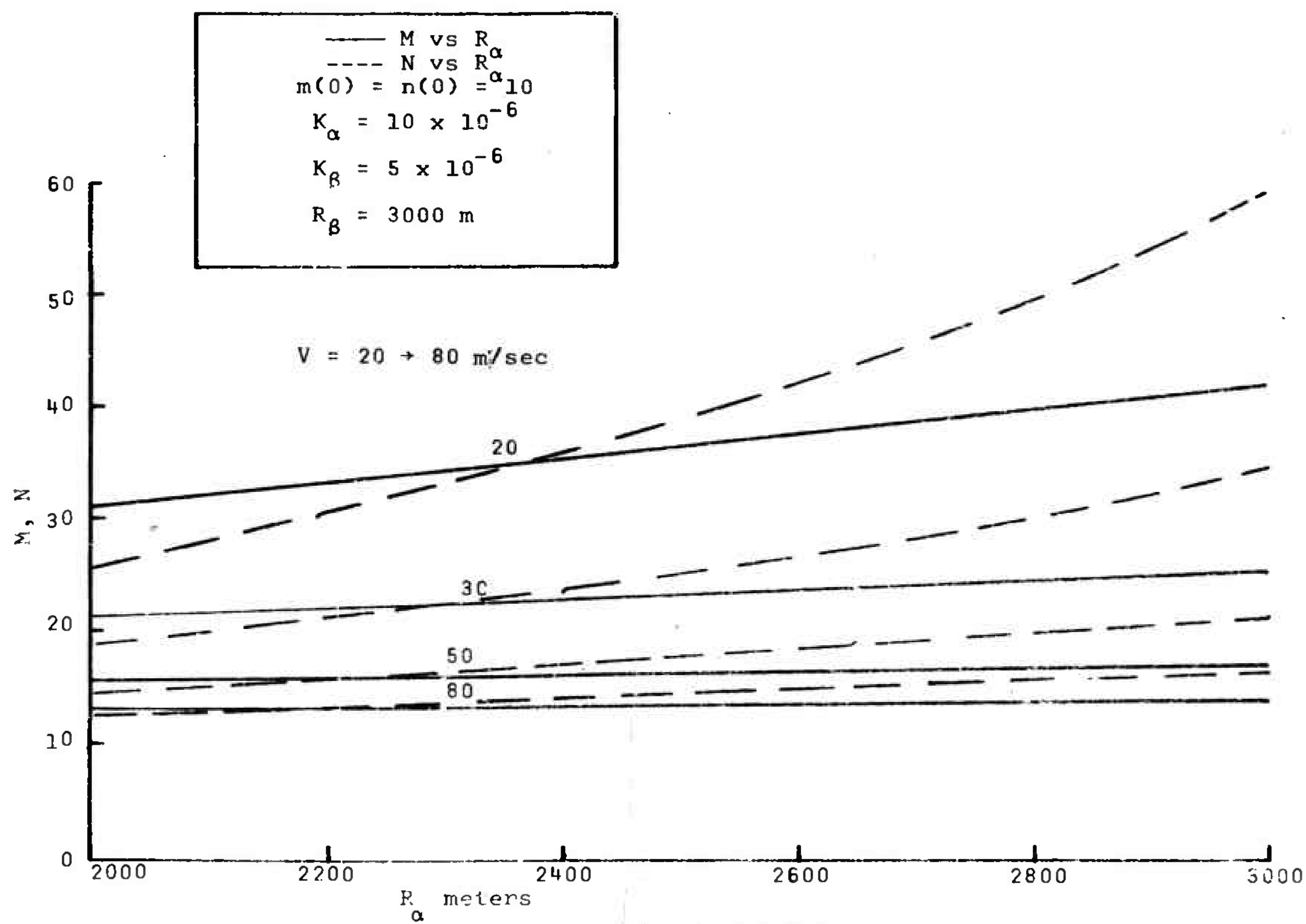

m(0) = n(0) = 10
K_α = 10 x 10^{-6}
K_β = 5 x 10^{-6}
R_β = 3000 m
—— M vs V
---- N vs V
R_α = 2000 → 3000 m
M, N
V(m/sec)
0
20
40
60
80
100
10
30
50
70
2
3

Figure 2.1.1.3



Figures 2.1.1.5 and 2.1.1.6



—— M vs R_α
---- N vs R_α
$m(0) = n(0) = 10$
$K_\alpha = 10 \times 10^{-6}$
$K_\beta = 5 \times 10^{-6}$
$R_\beta = 3000$ m
V = 20 → 80 m/sec
20
30
50
80
M, N
60
50
40
30
20
10
0
2000
2200
2400
2600
2800
3000
R_α meters

Figure 2.1.3.1

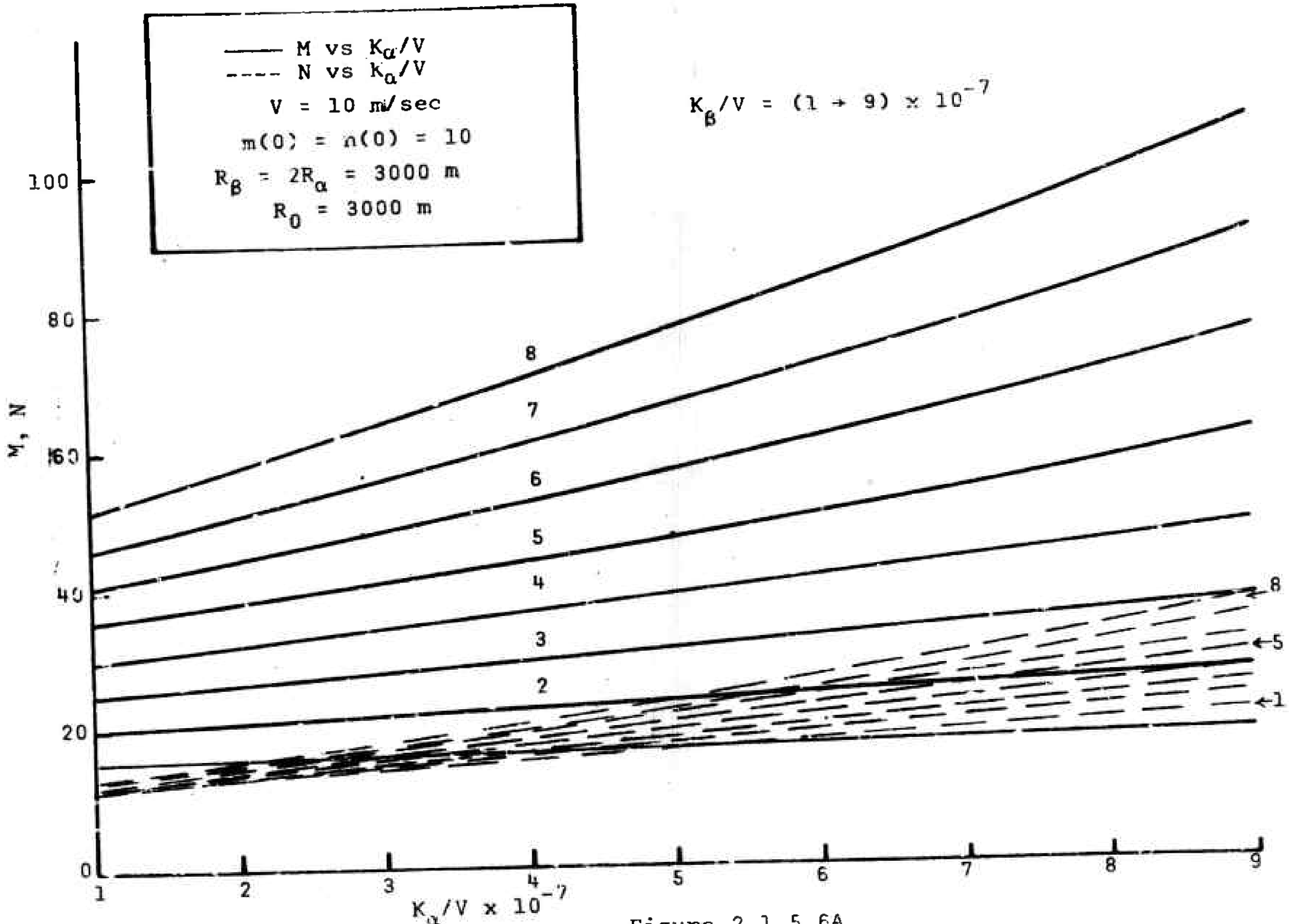


Figure 2.1.5.6A

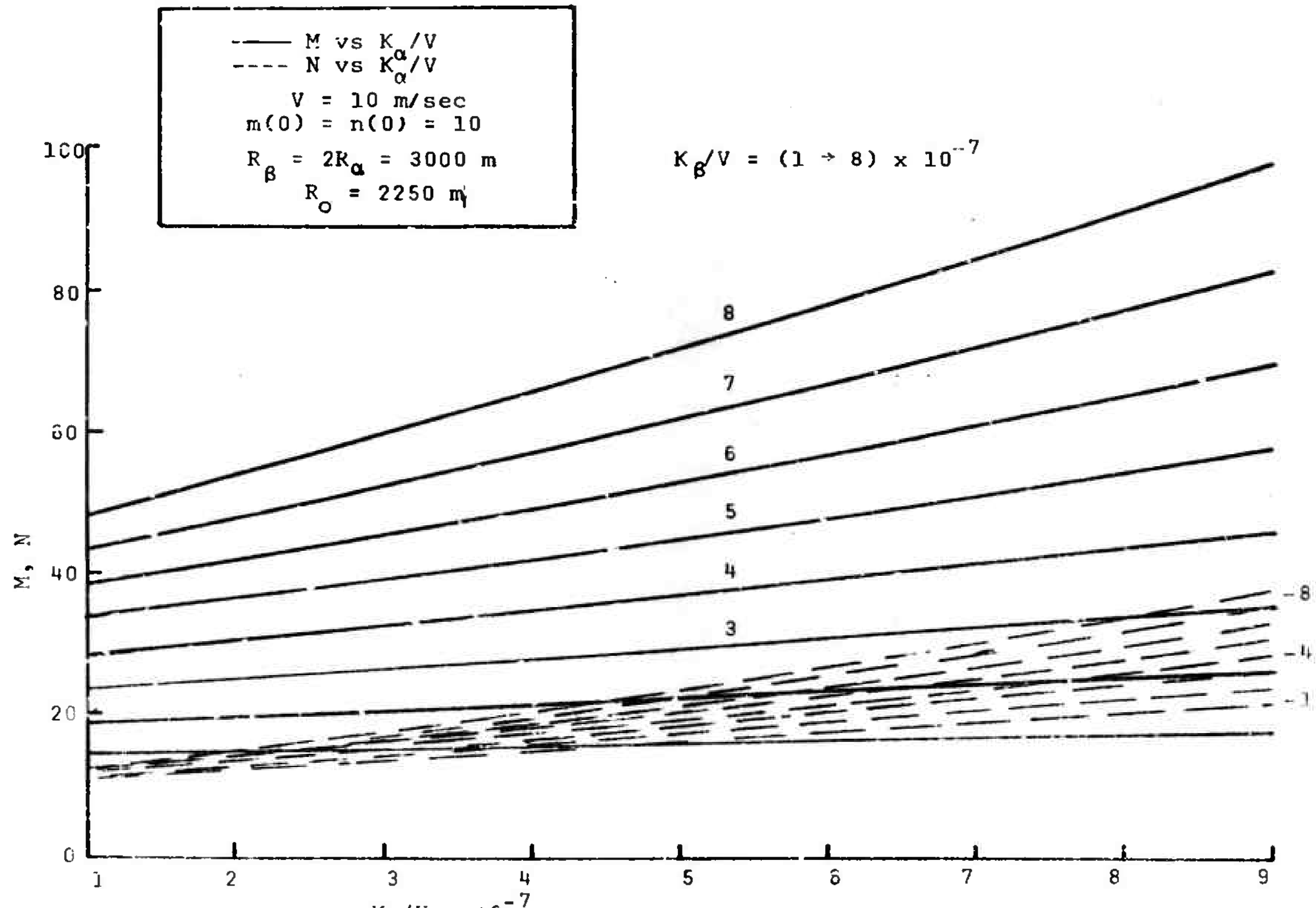


Figure 2.1.5.6B

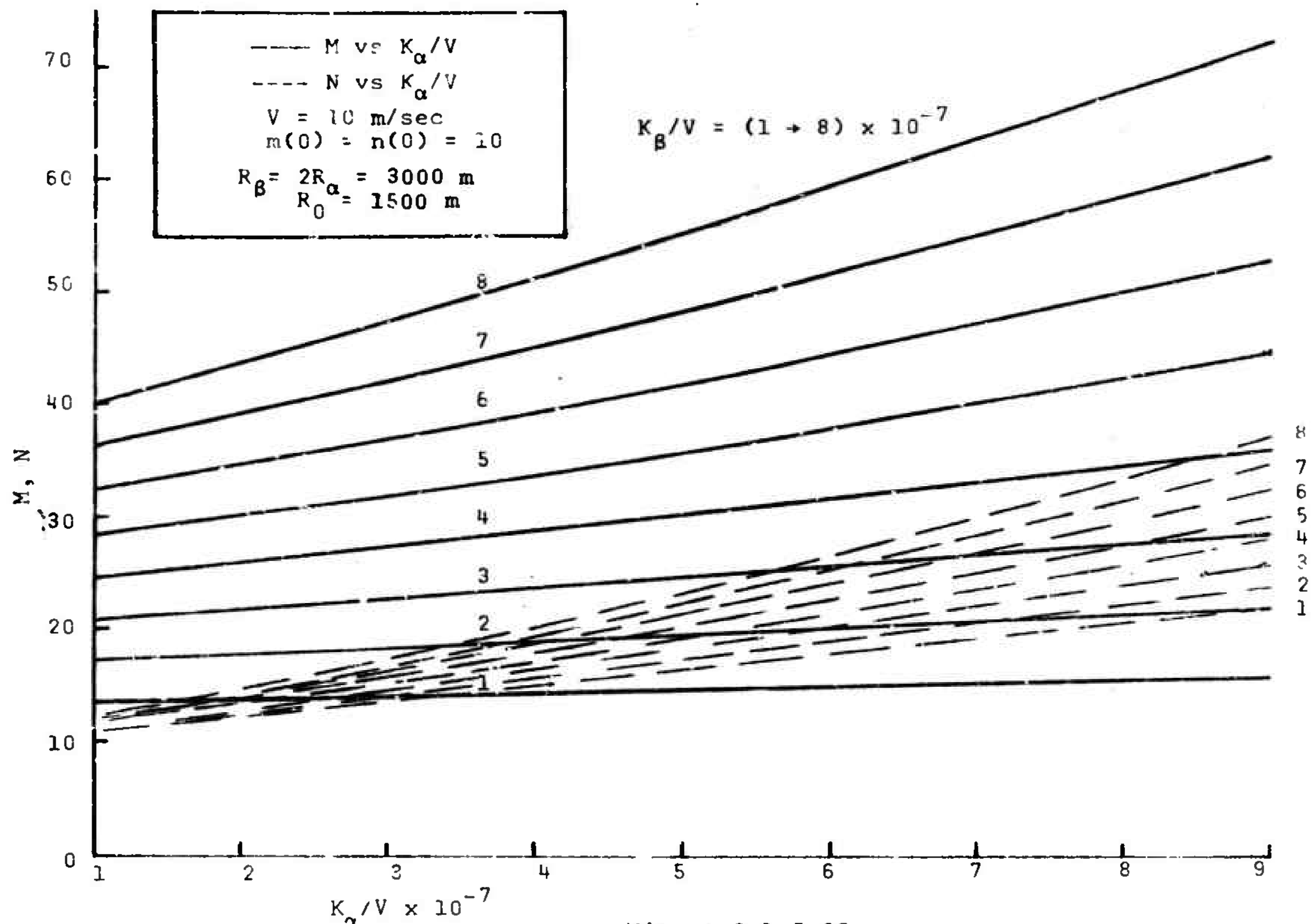


Figure 2.1.5.6C

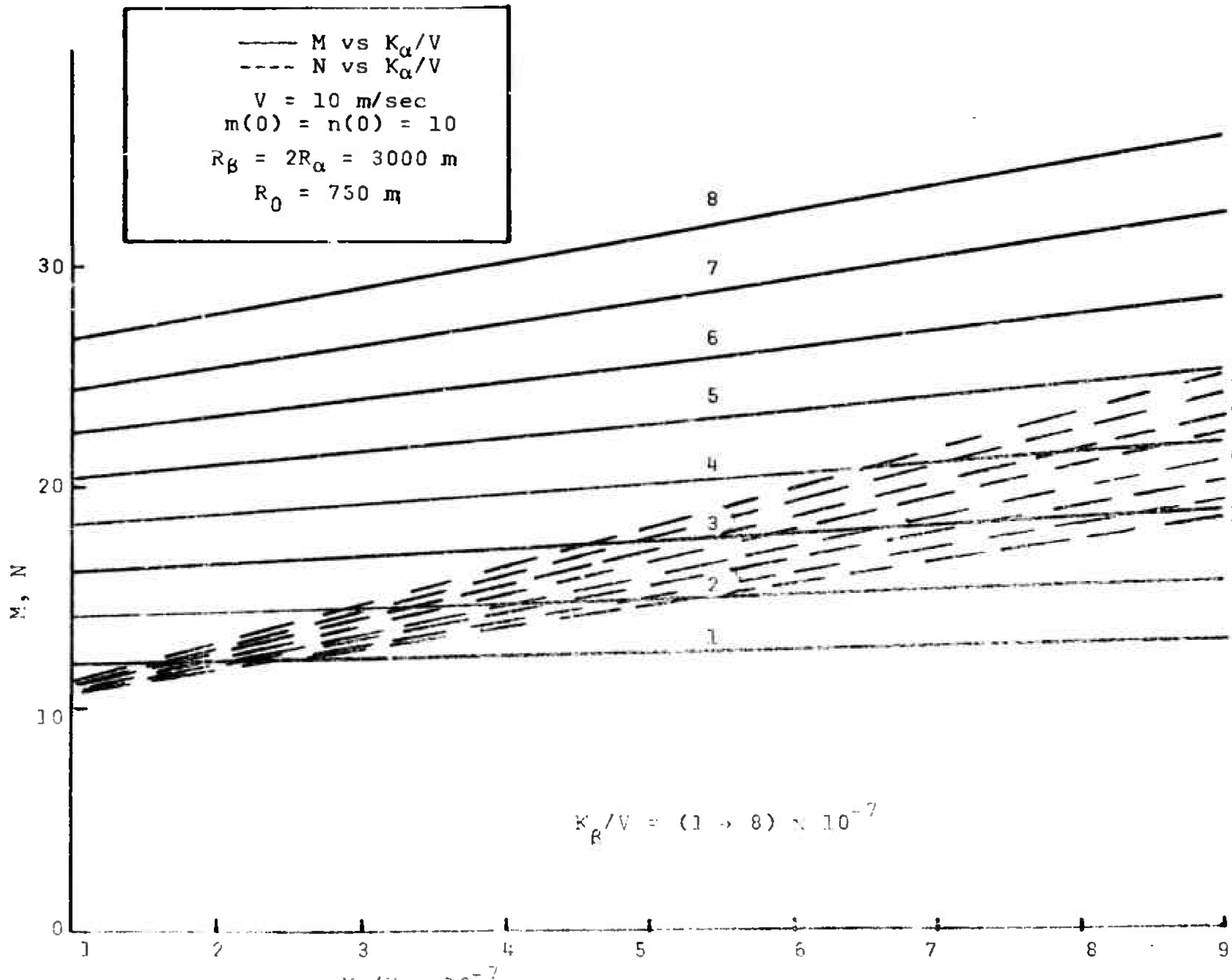


Figure 2.1.5.6D

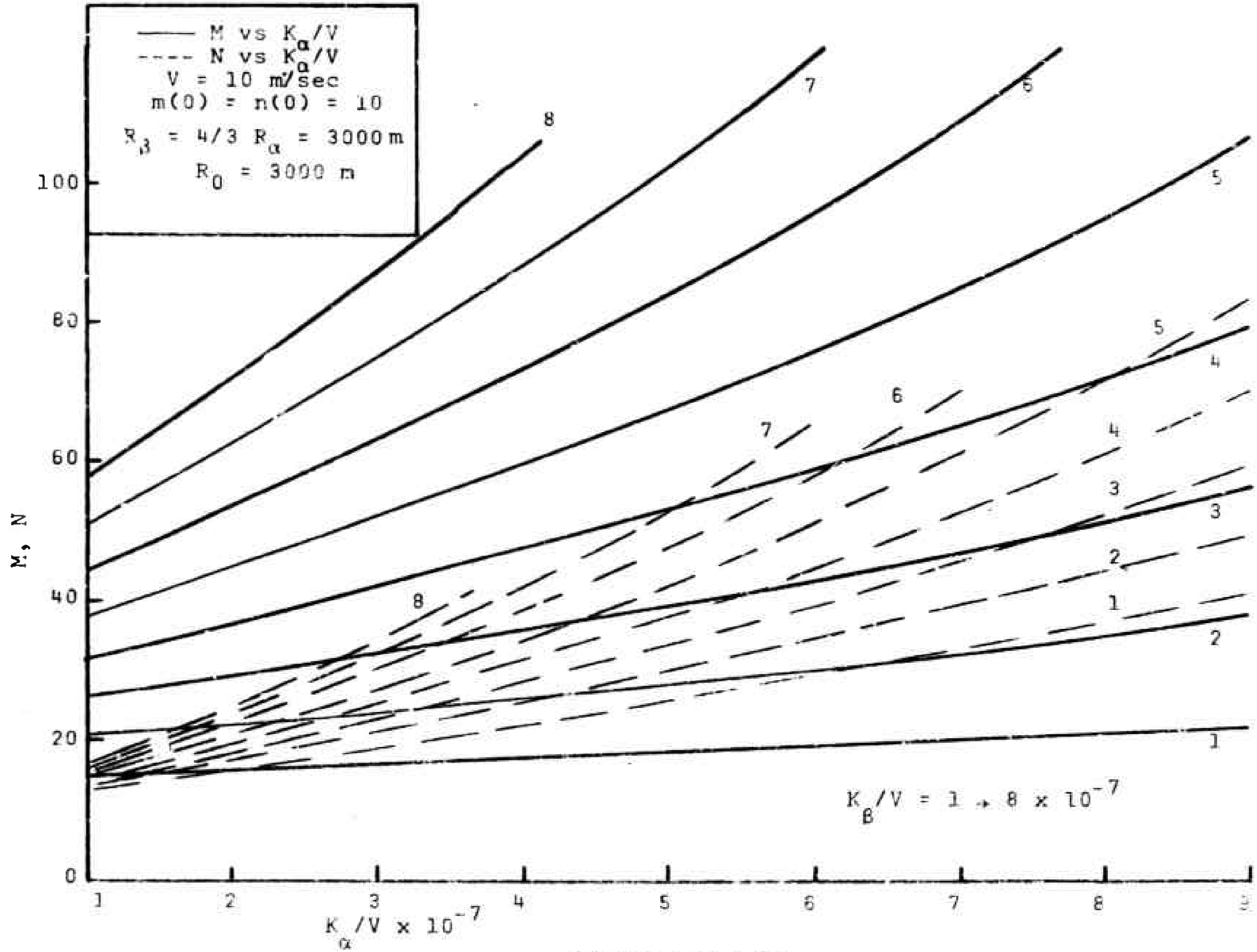


Figure 2.1.5.6E

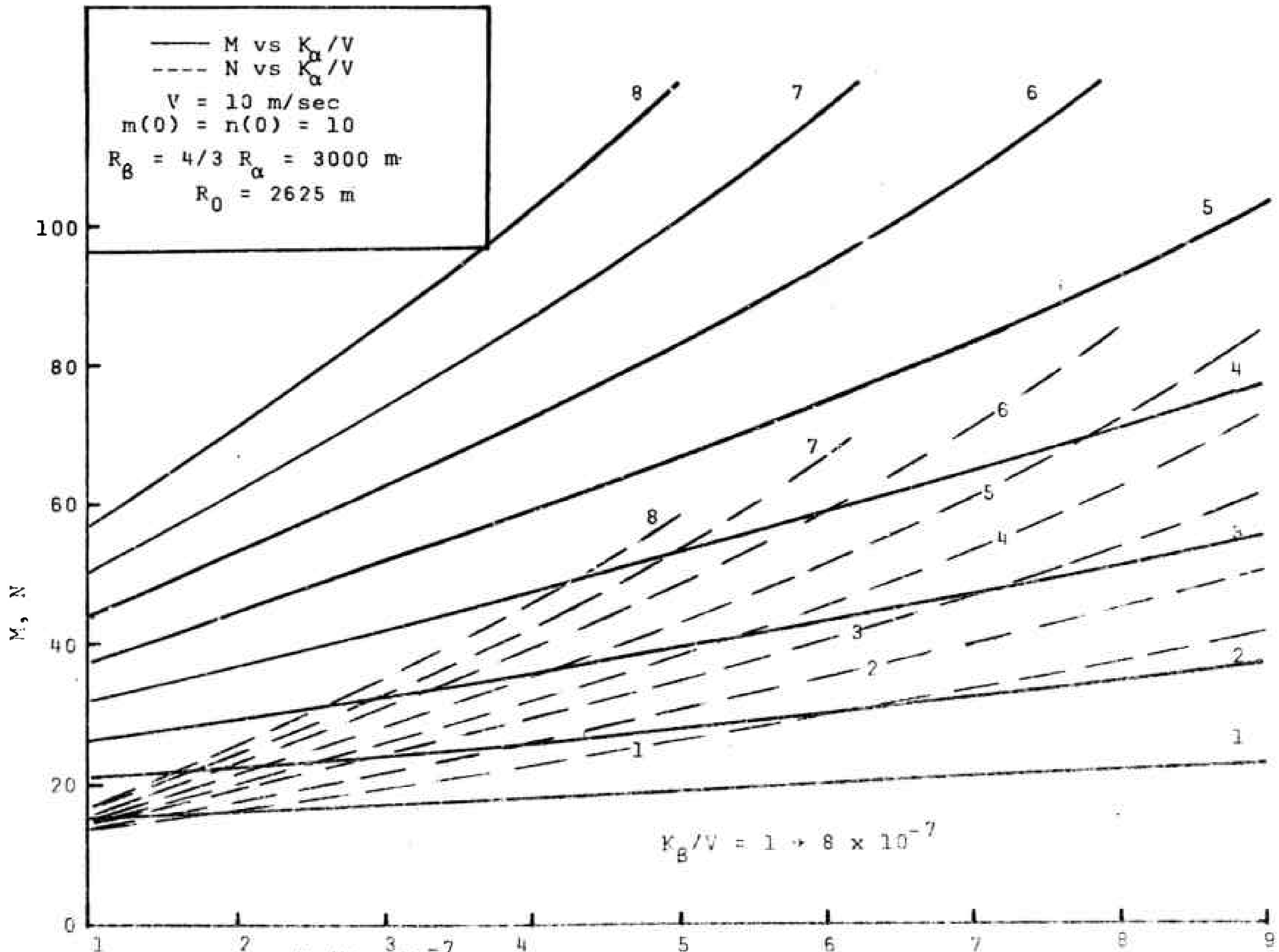


Figure 2.1.5.57

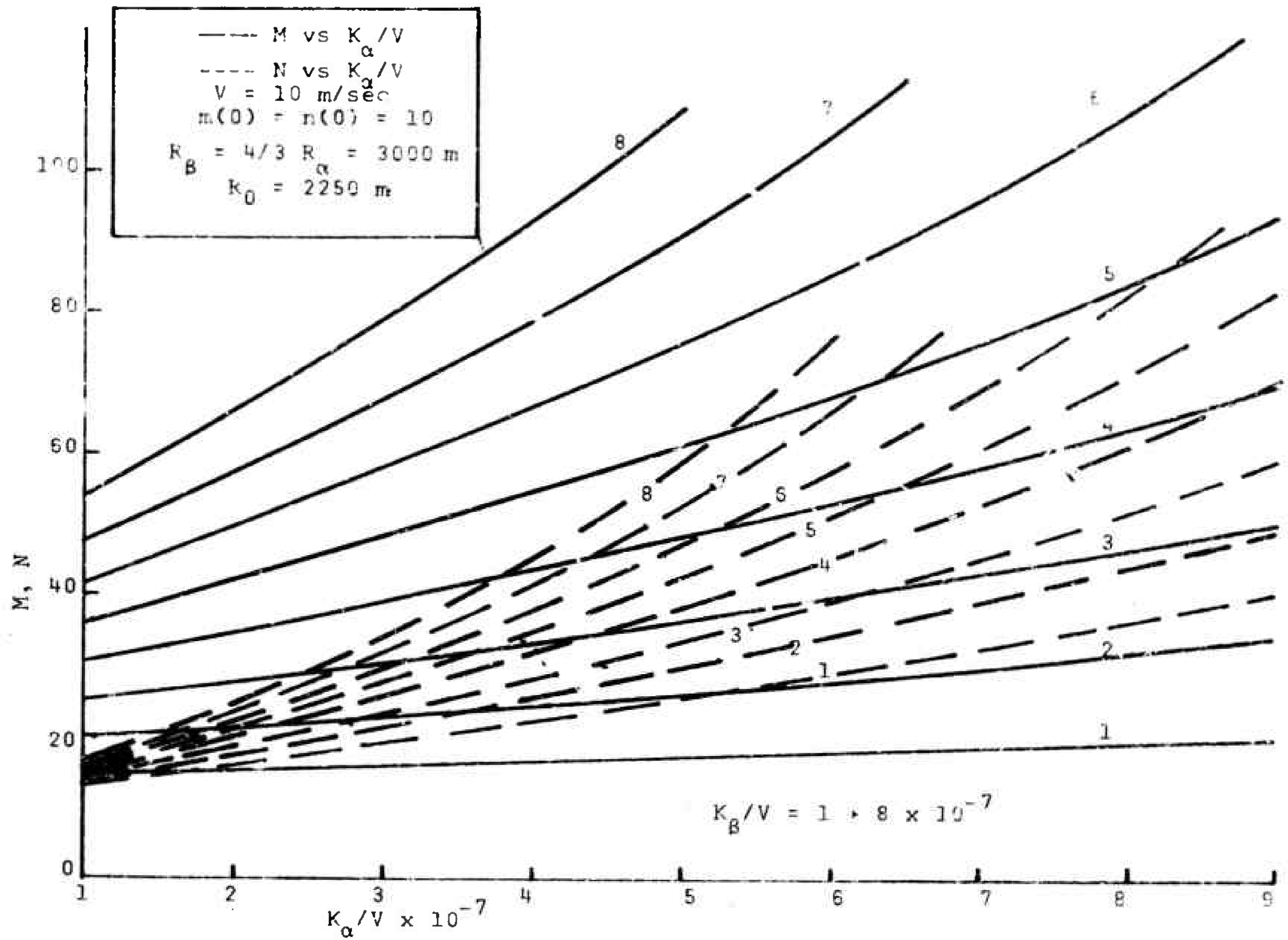


Figure 2.1.5.6G

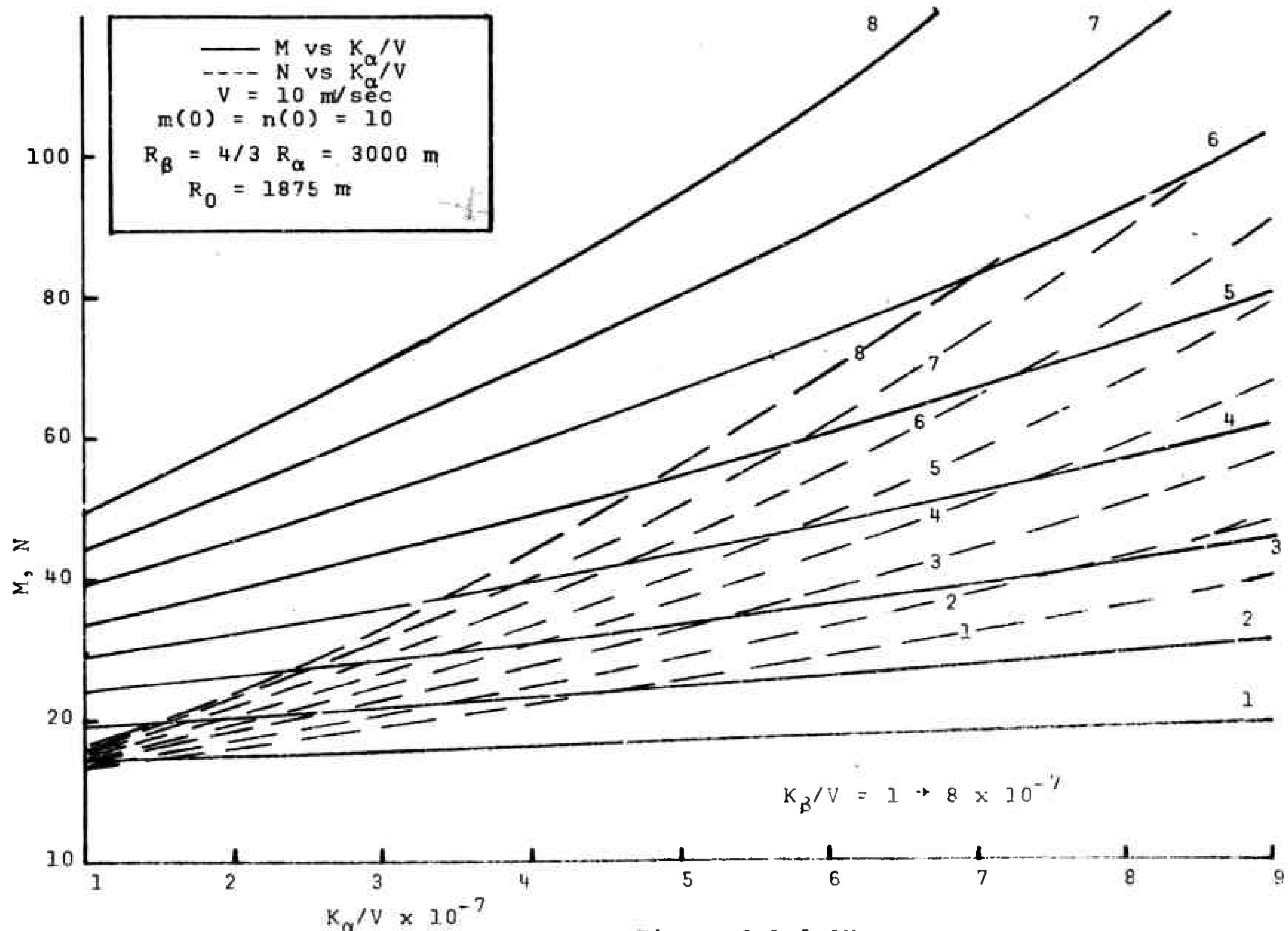


Figure 2.1.5.6H

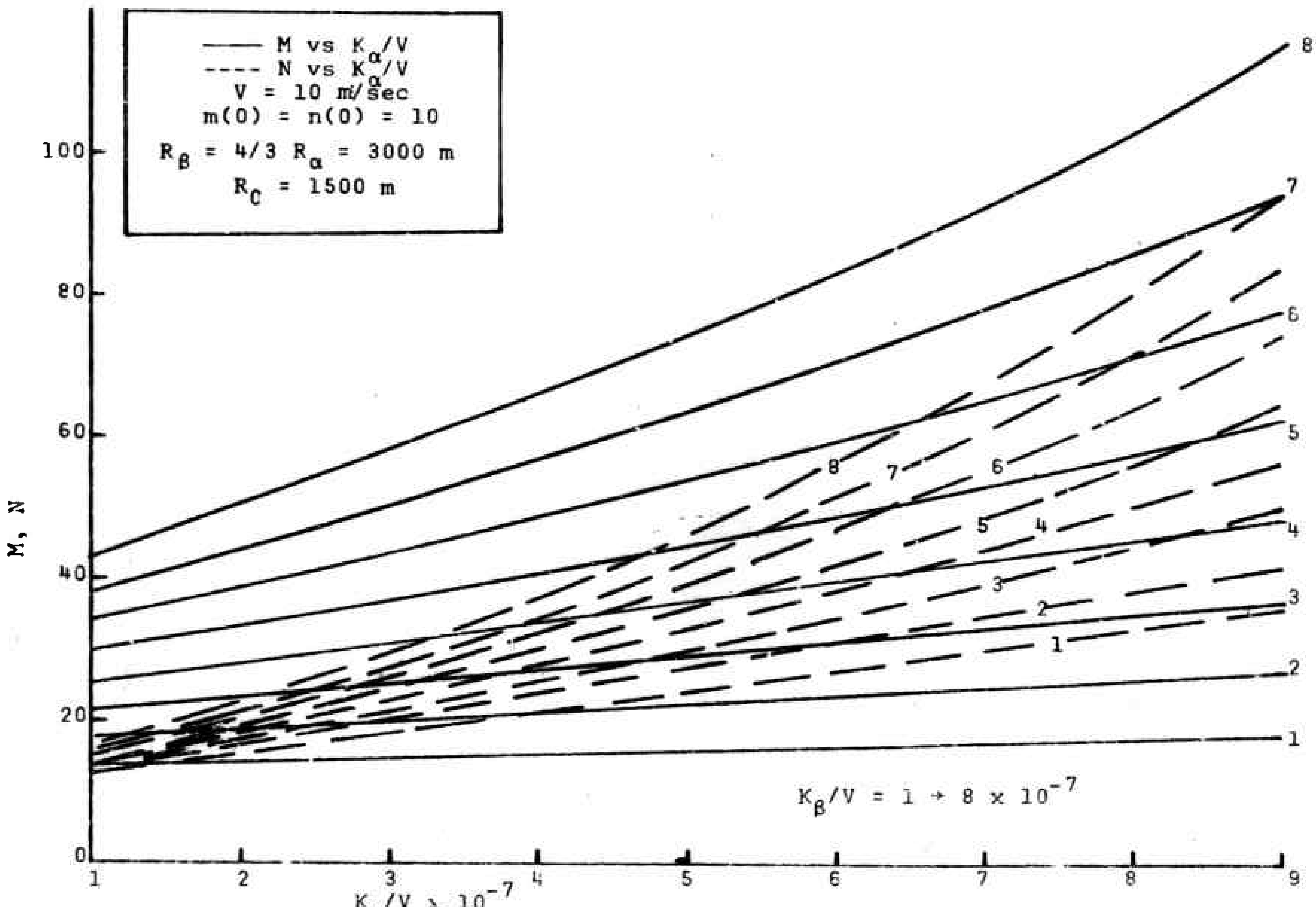


Figure 2.1.5.6I

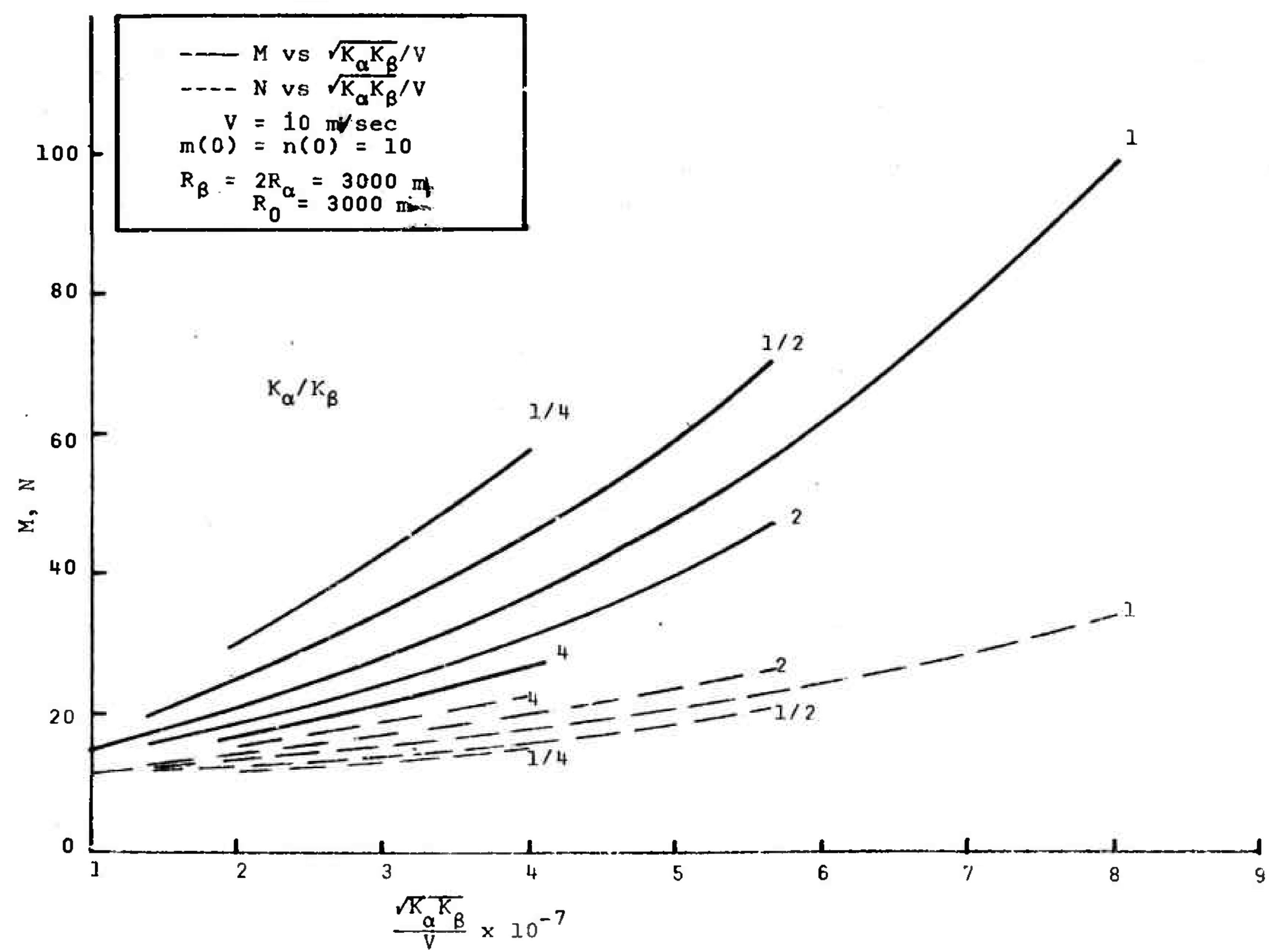


Figure 2.1.7.8A

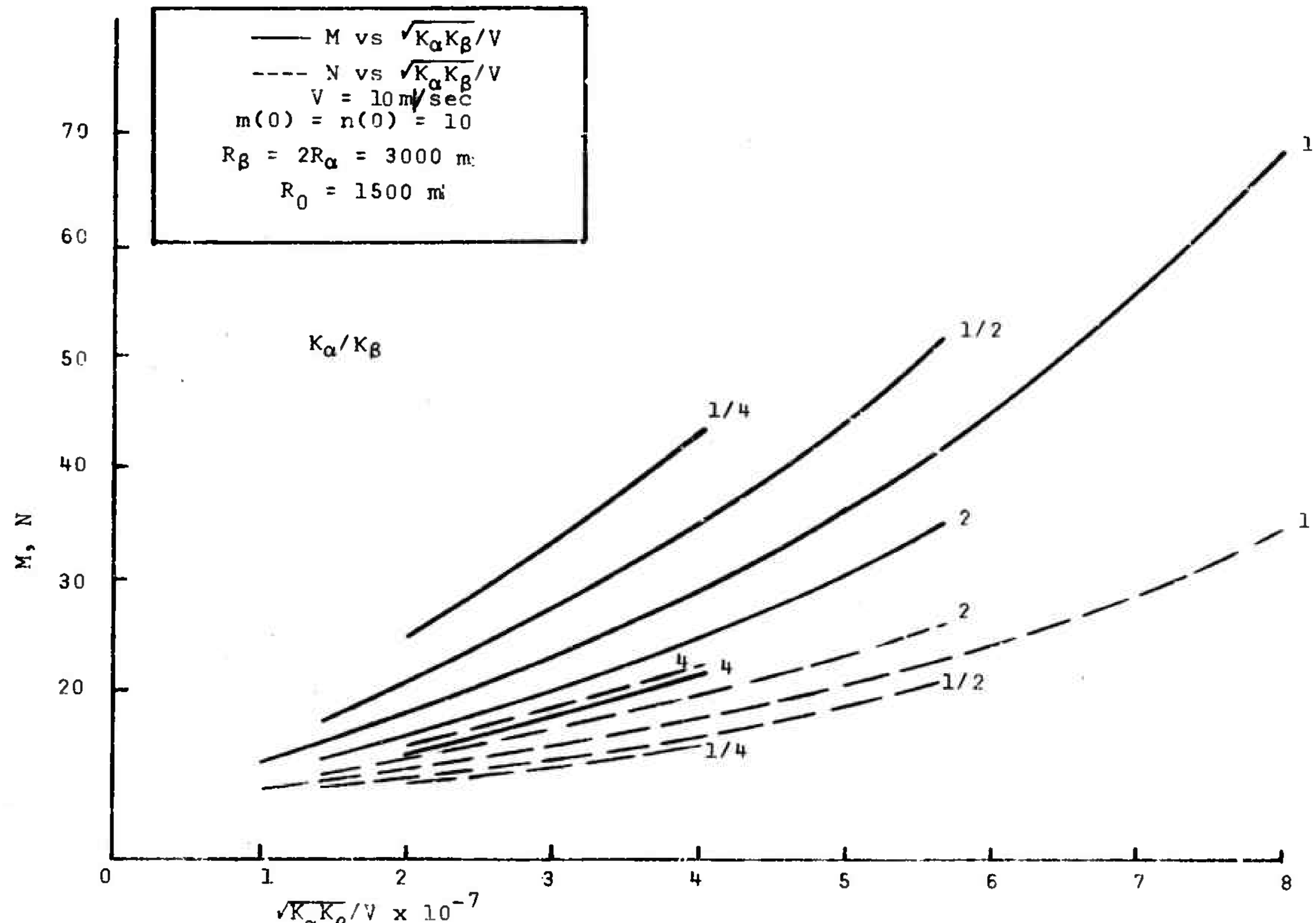


Figure 2.1.7.8B

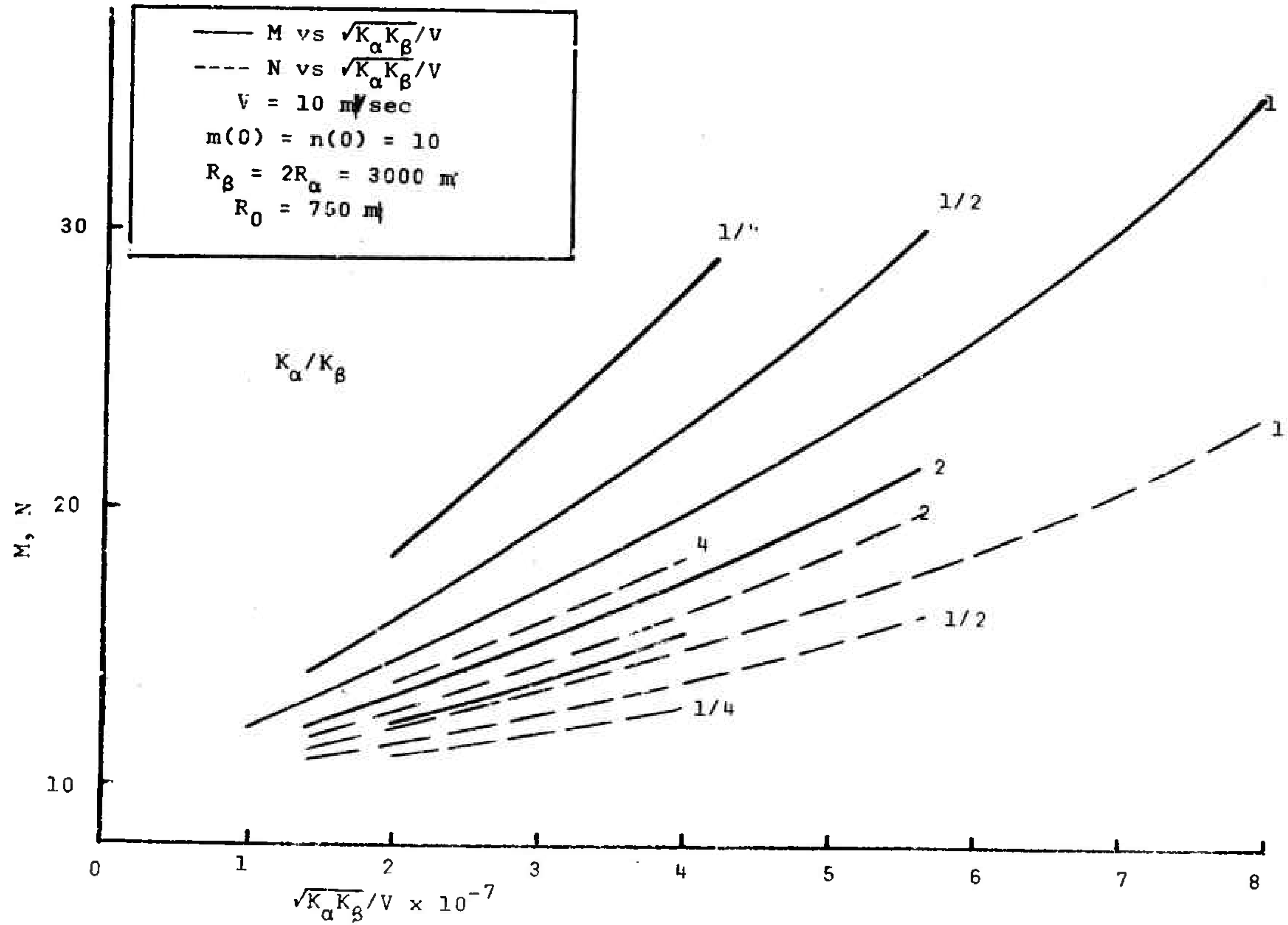
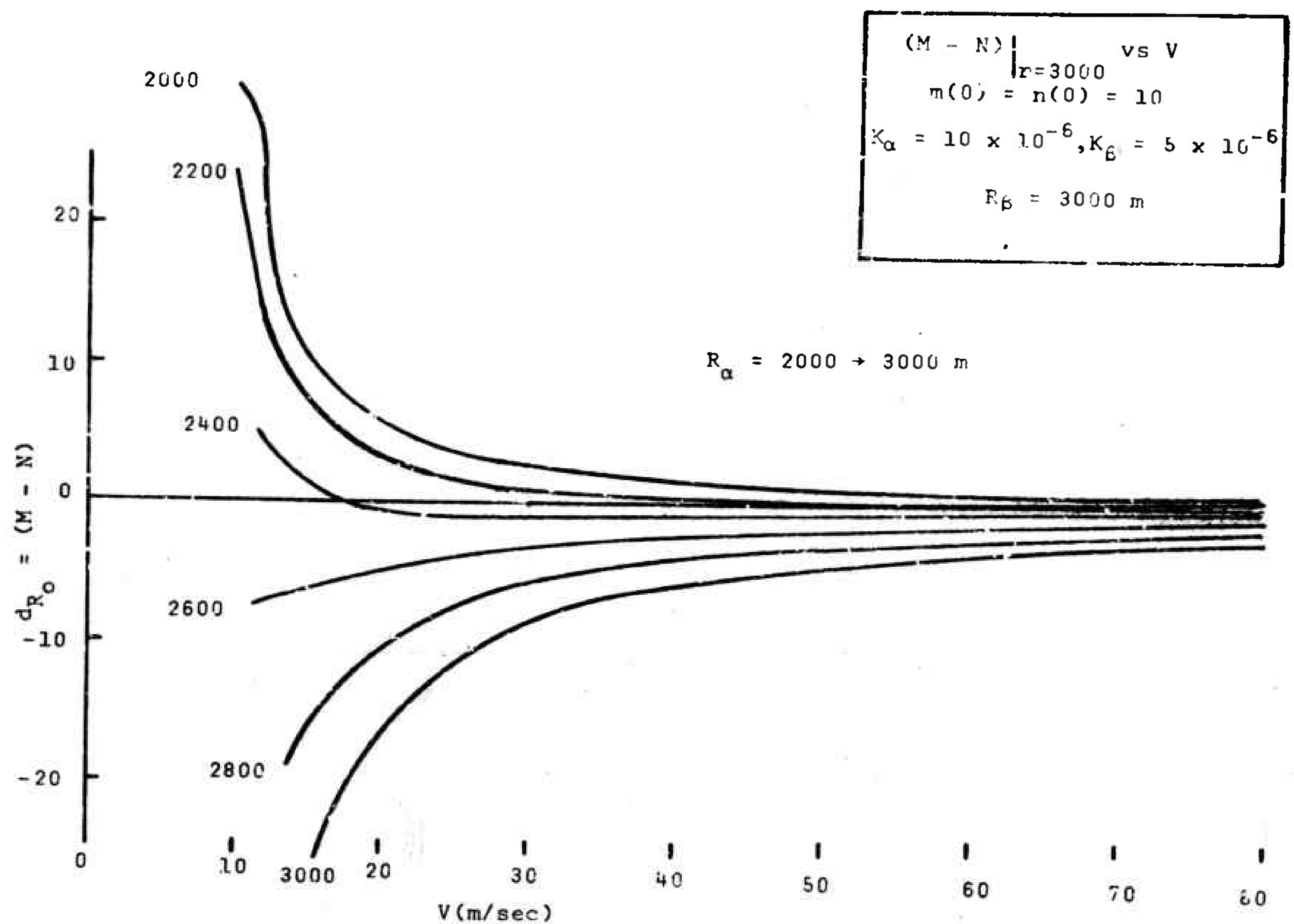


Figure 2.1.7.8C



$(M - N)\big|_{r=3000}$ vs V
$m(0) = n(0) = 10$
$K_\alpha = 10 \times 10^{-6}, K_\beta = 5 \times 10^{-6}$
$R_\beta = 3000$ m
$R_\alpha = 2000 \rightarrow 3000$ m
2000
2200
2400
2600
2800
3000
20
10
0
-10
-20
$d_{R_o} = (M - N)$
0
10
20
30
40
50
60
70
80
V(m/sec)

Figure 2.2.1.?

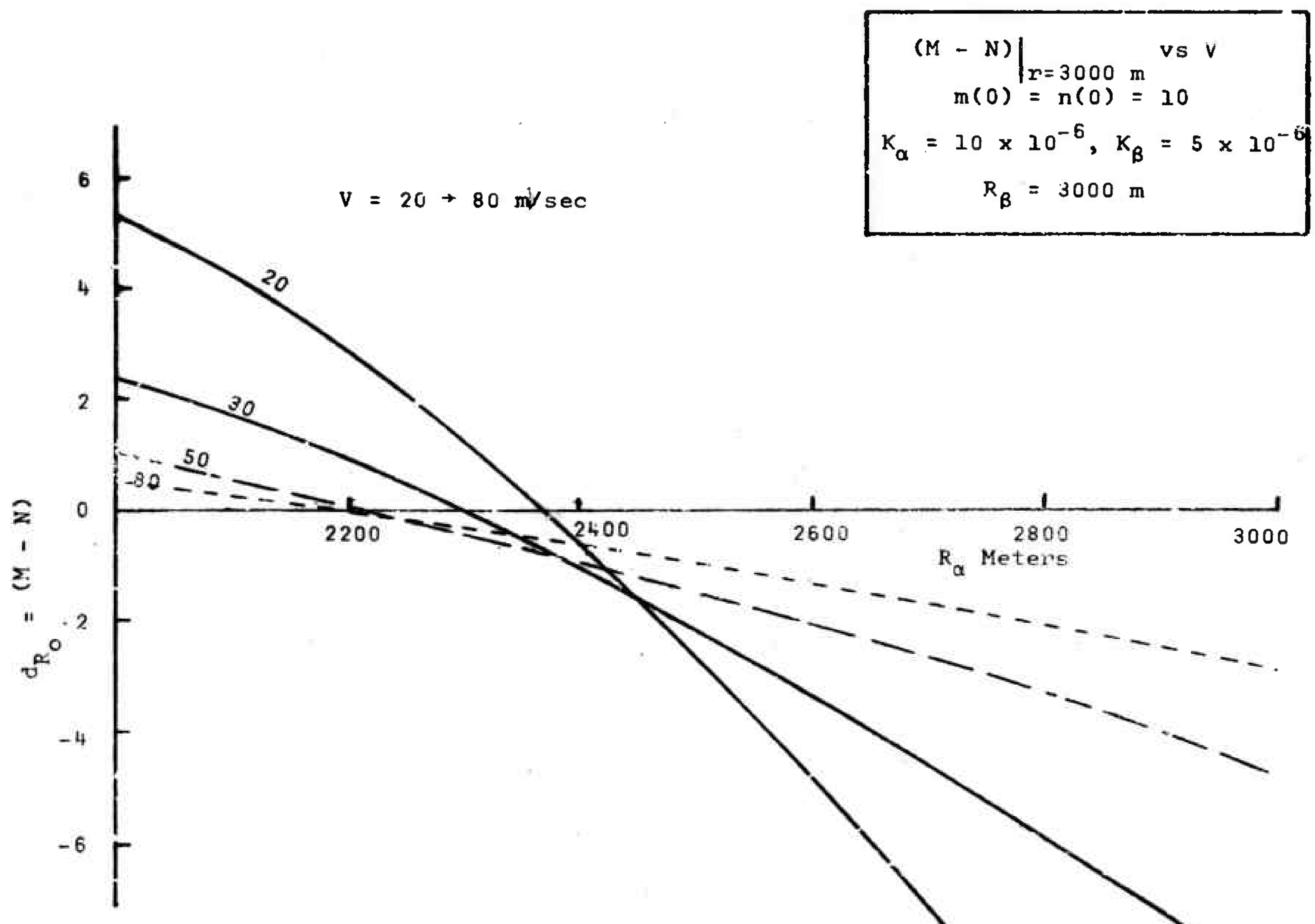

(M - N)|r=3000 m vs V
m(0) = n(0) = 10
Kα = 10 x 10^-6, Kβ = 5 x 10^-6
Rβ = 3000 m
V = 20 → 80 m/sec
20
30
50
80
6
4
2
0
-2
-4
-6
2200
2400
2600
2800
3000
Rα Meters
dRo = (M - N)

Figure 2.2.3.1

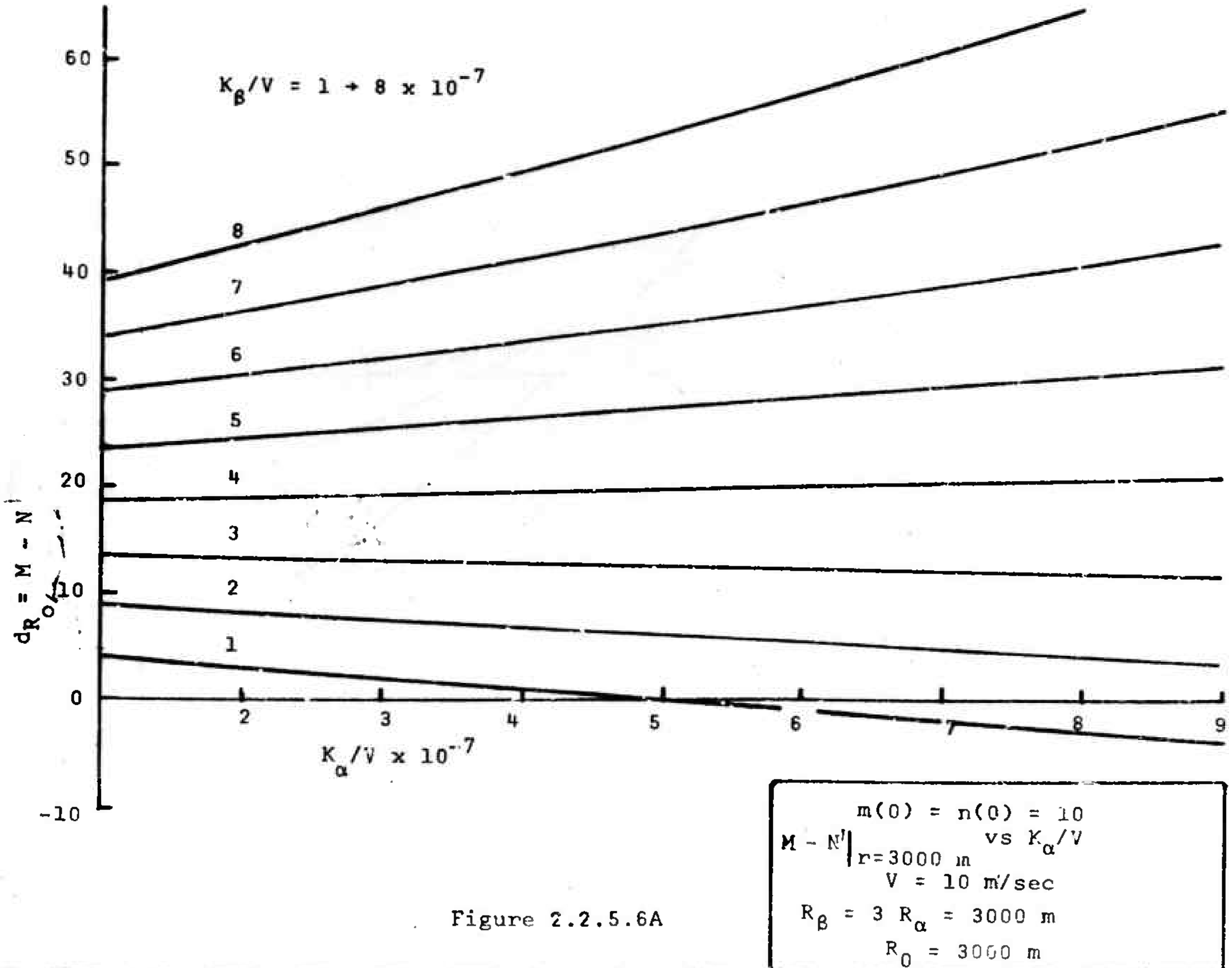


Figure 2.2.5.6A

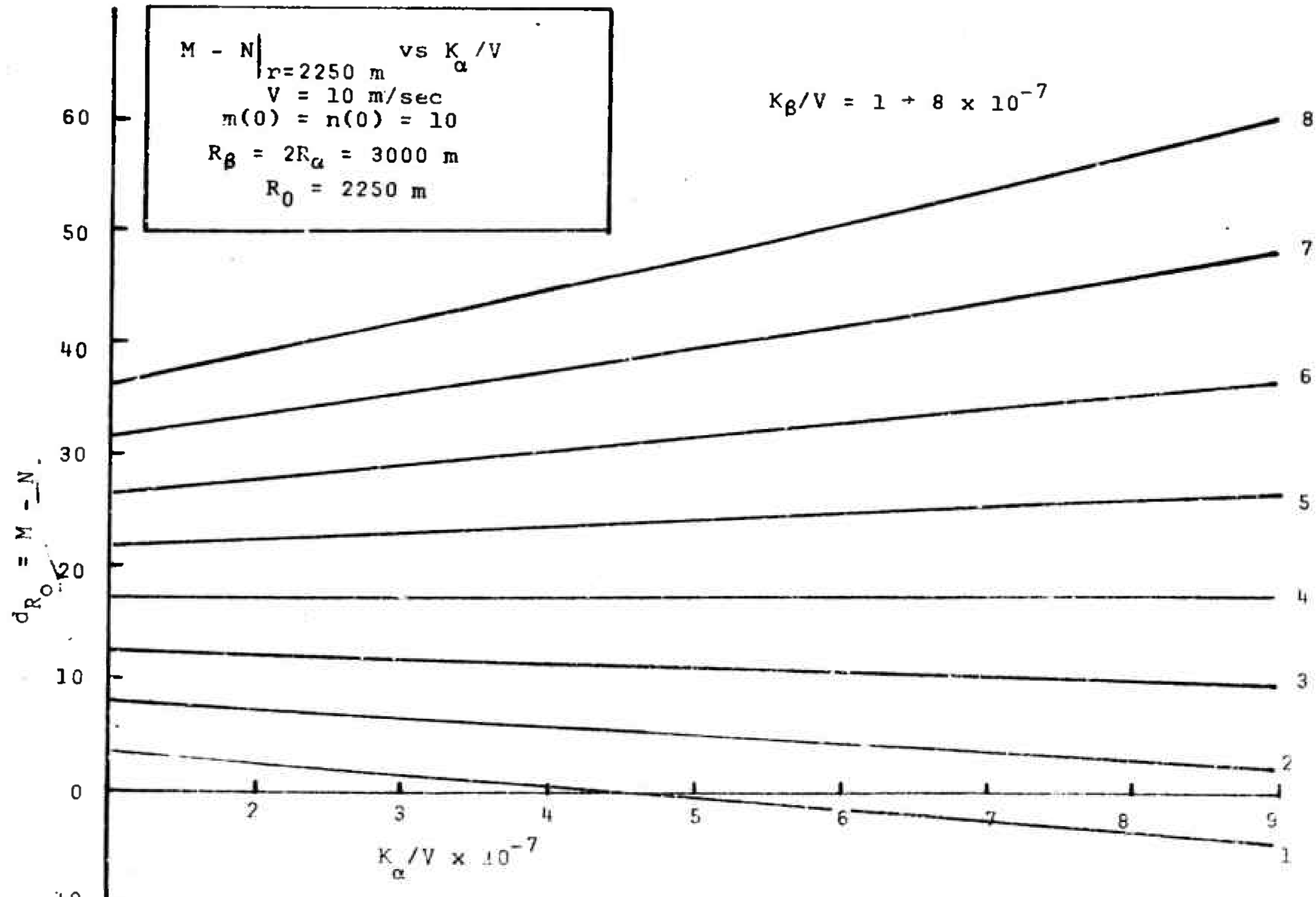


Figure 2.2.5.6B

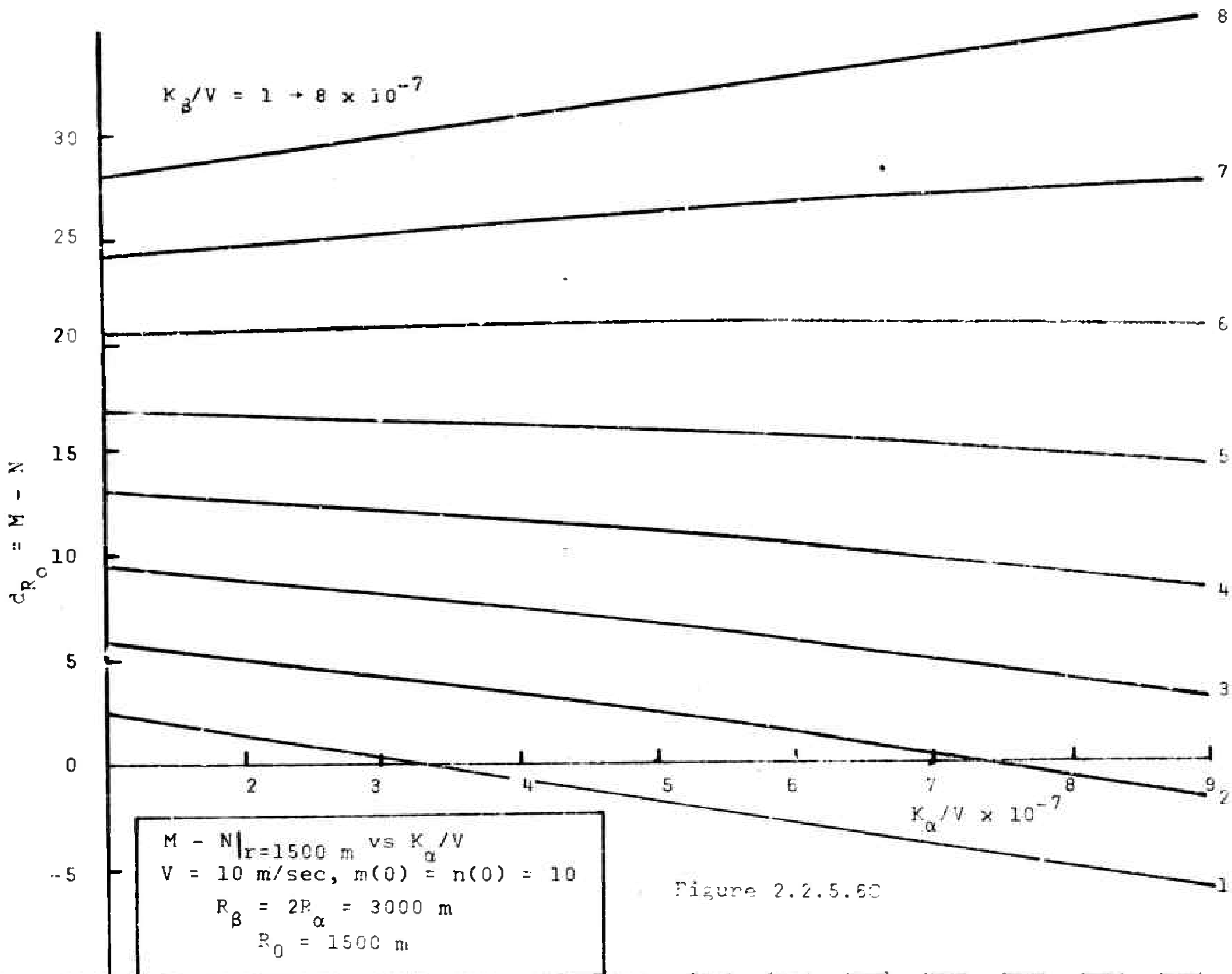


Figure 2.2.5.60

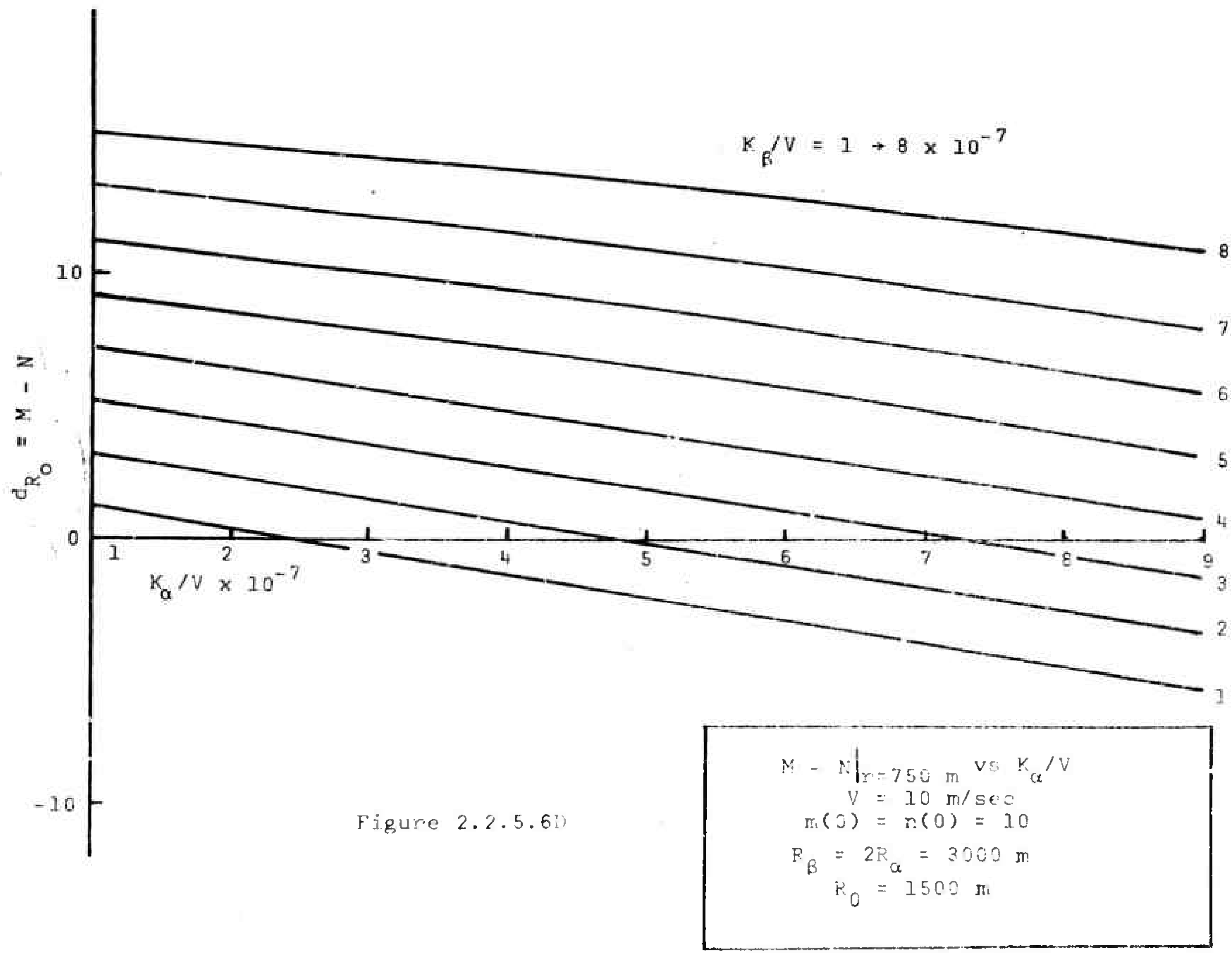


Figure 2.2.5.6D

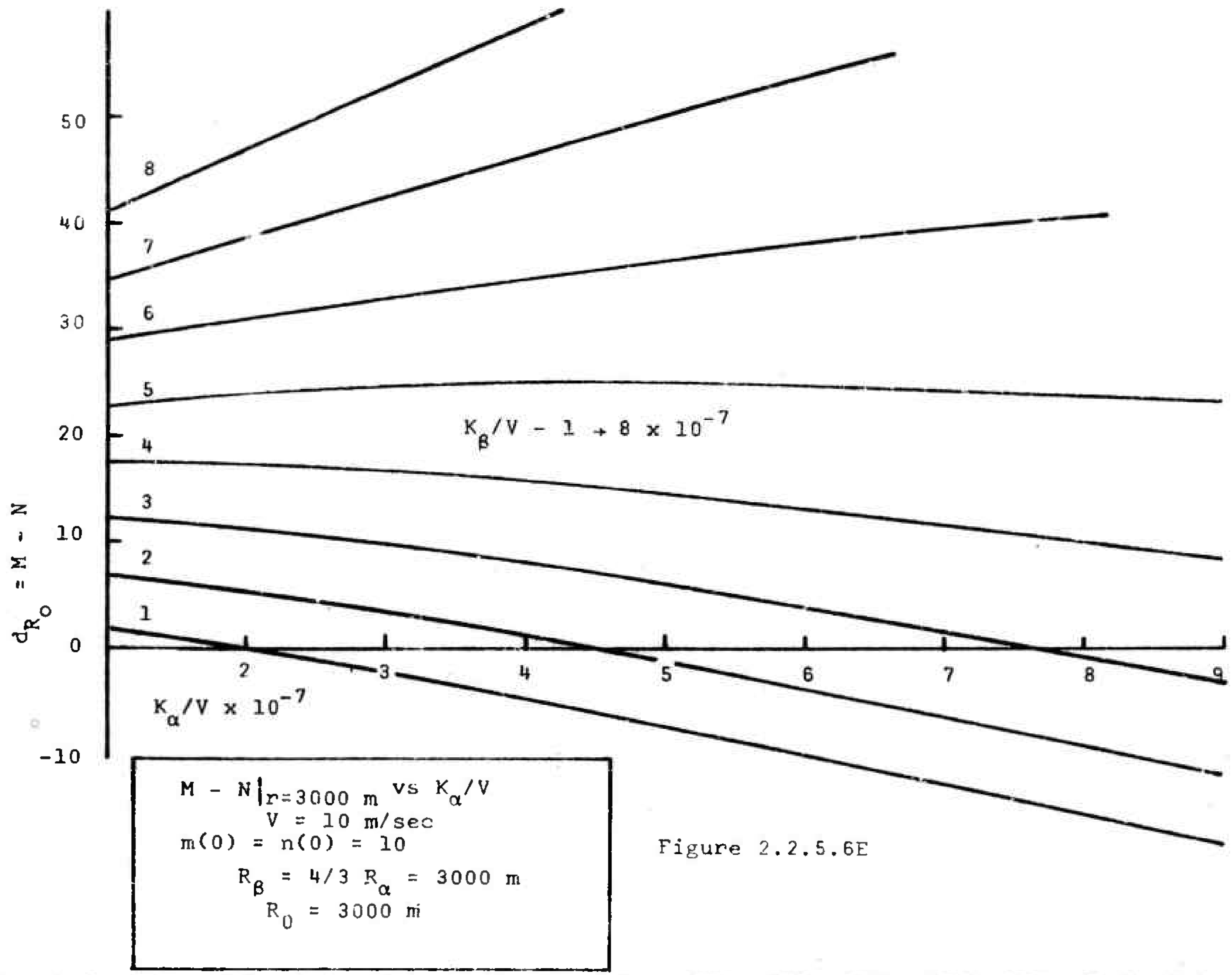


Figure 2.2.5.6E

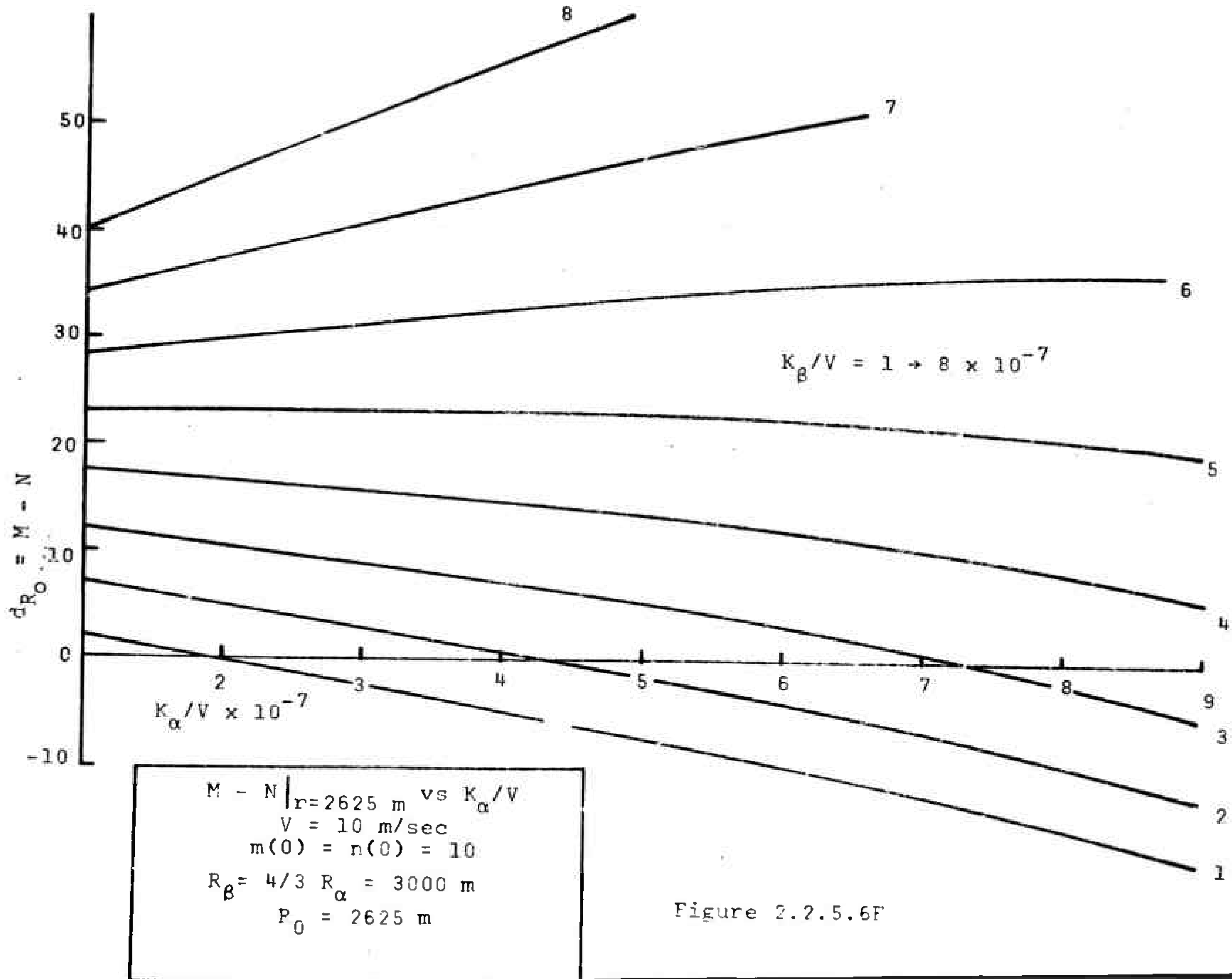


Figure 2.2.5.6F

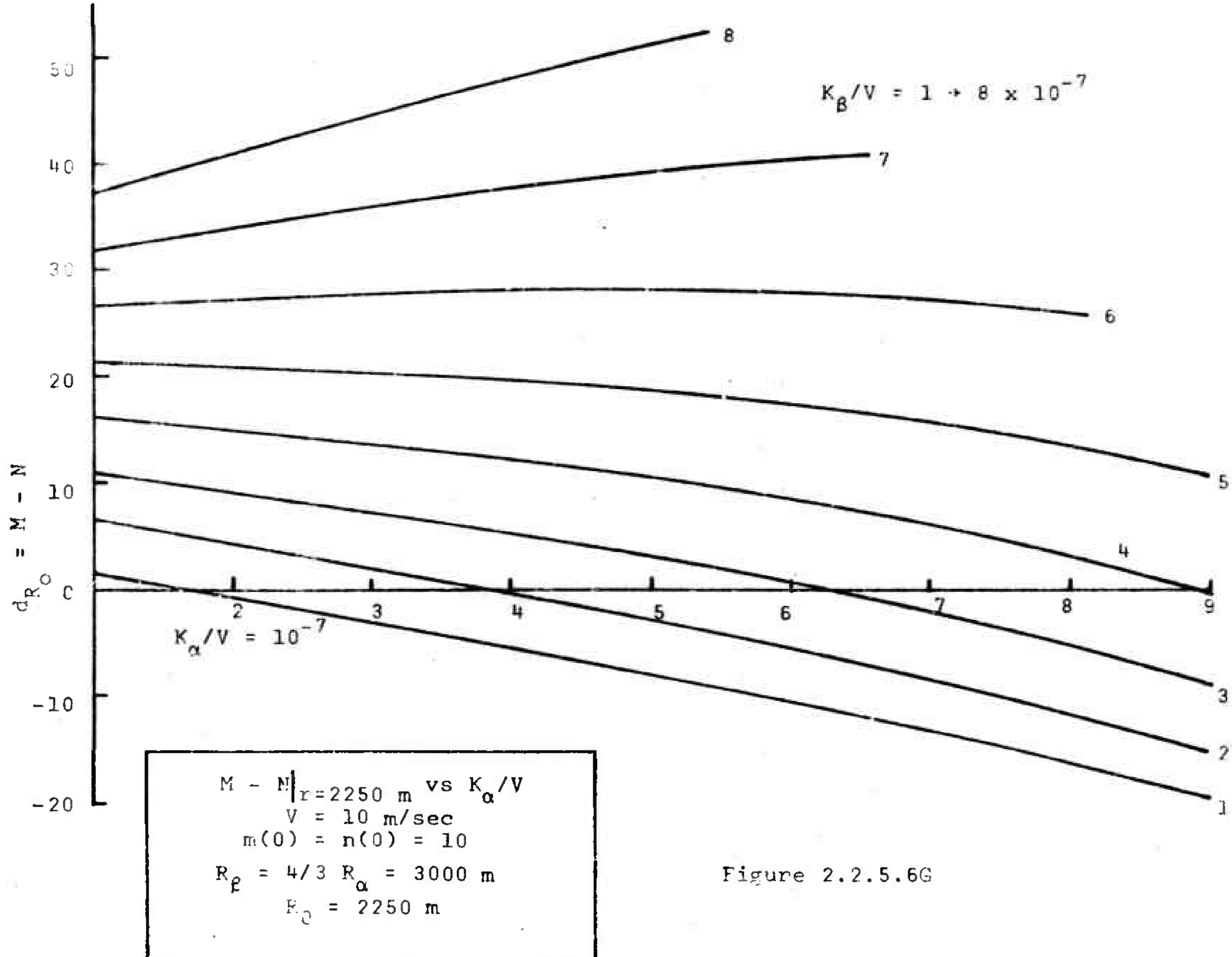


Figure 2.2.5.6G

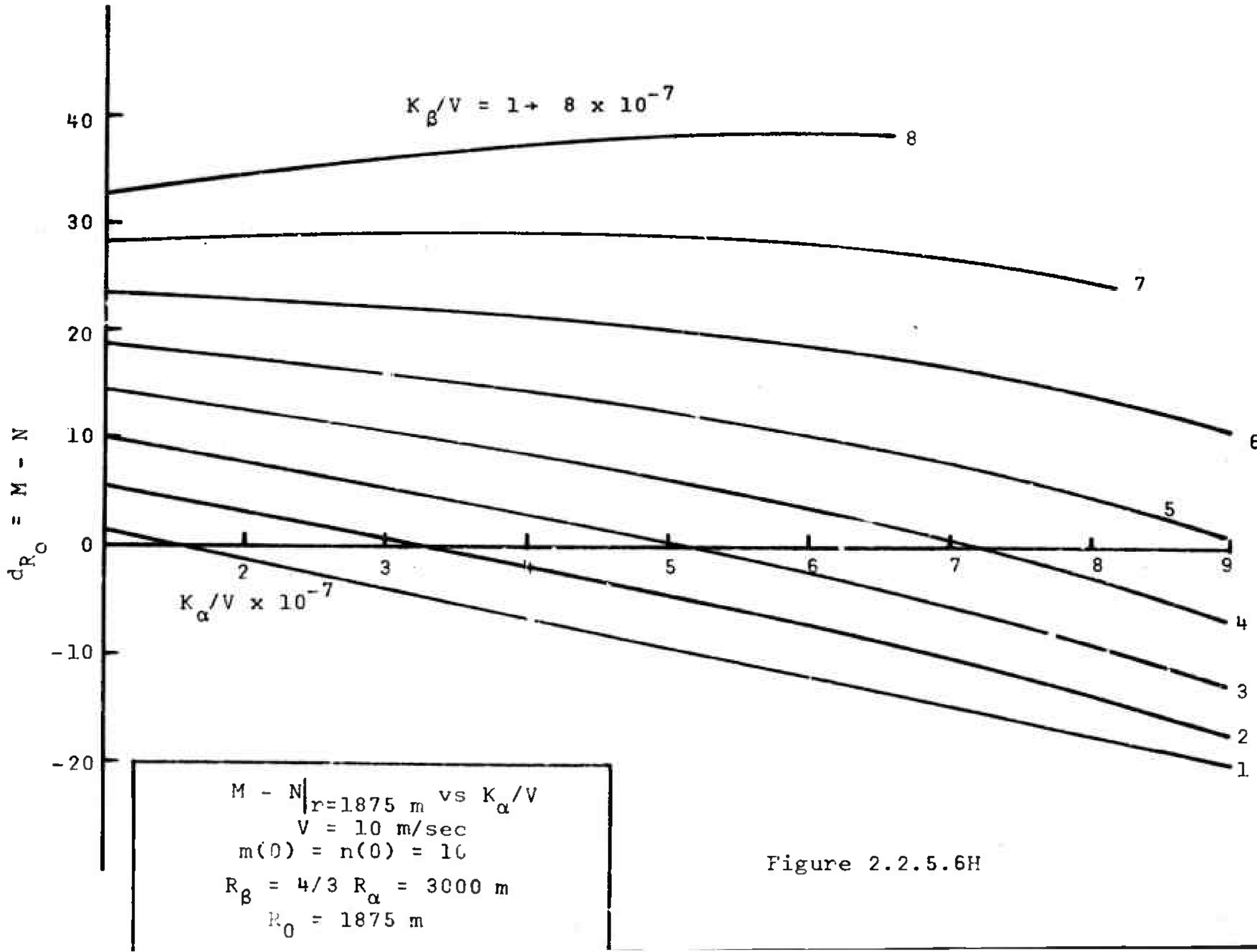


Figure 2.2.5.6H

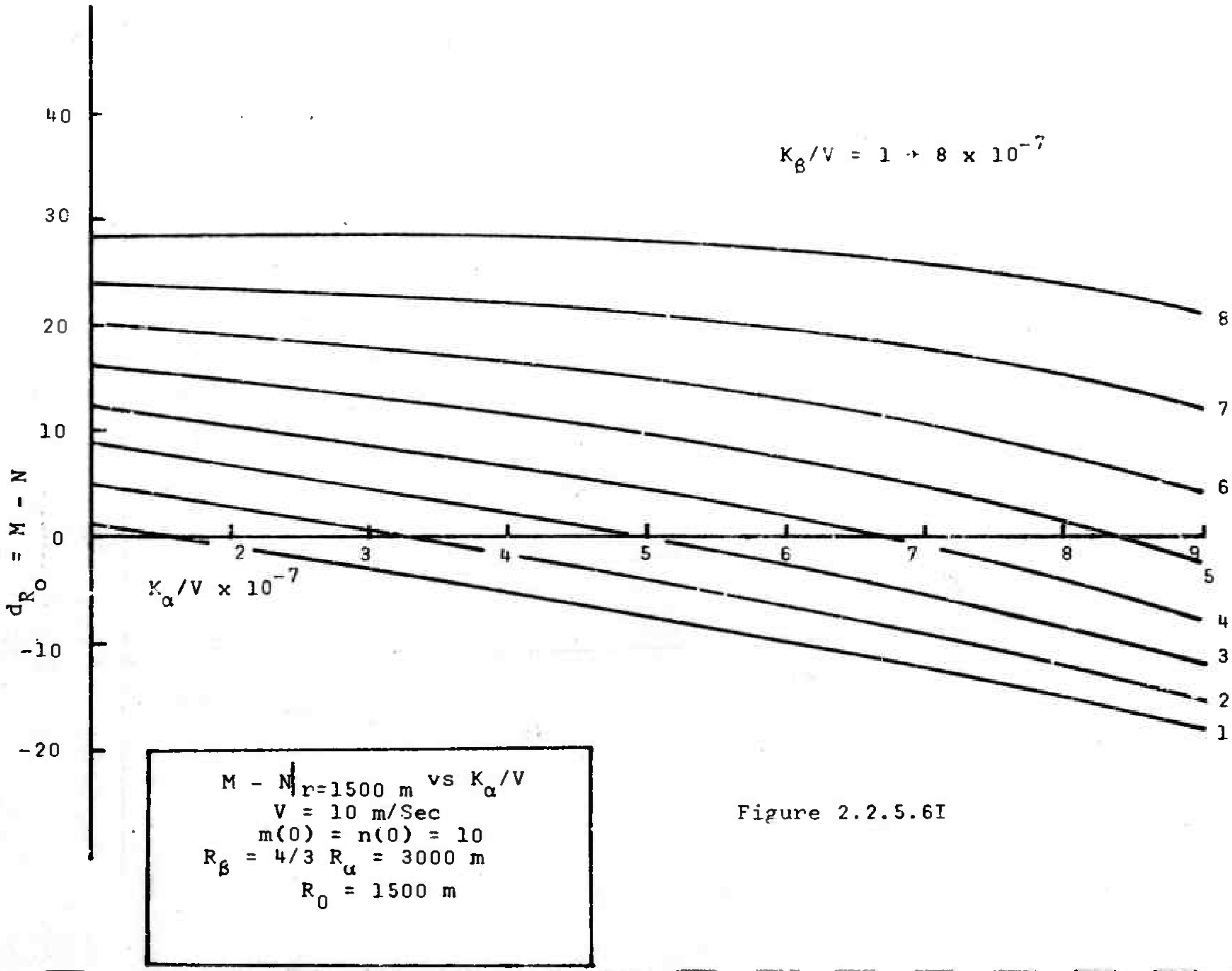


Figure 2.2.5.6I

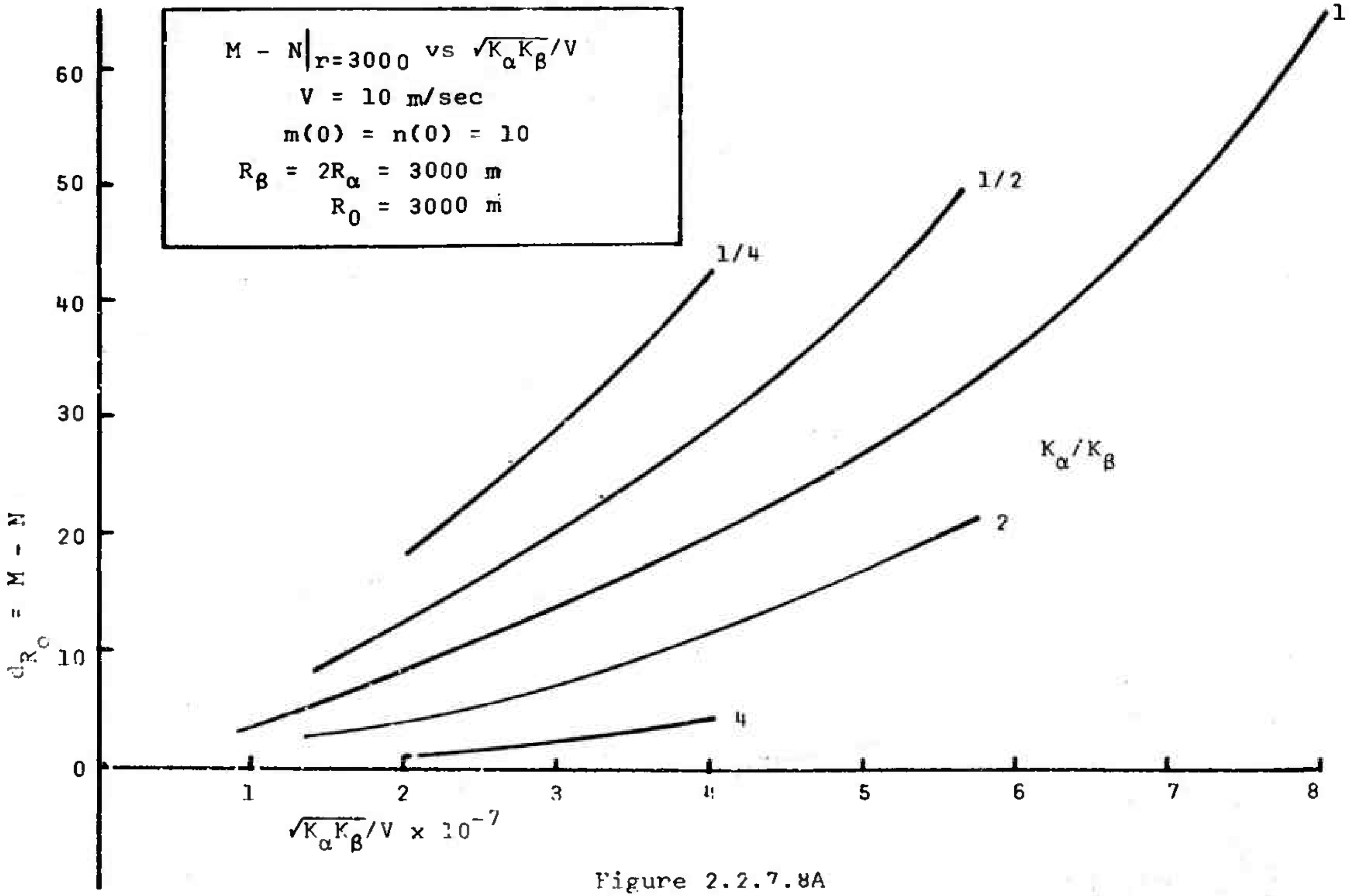


Figure 2.2.7.8A

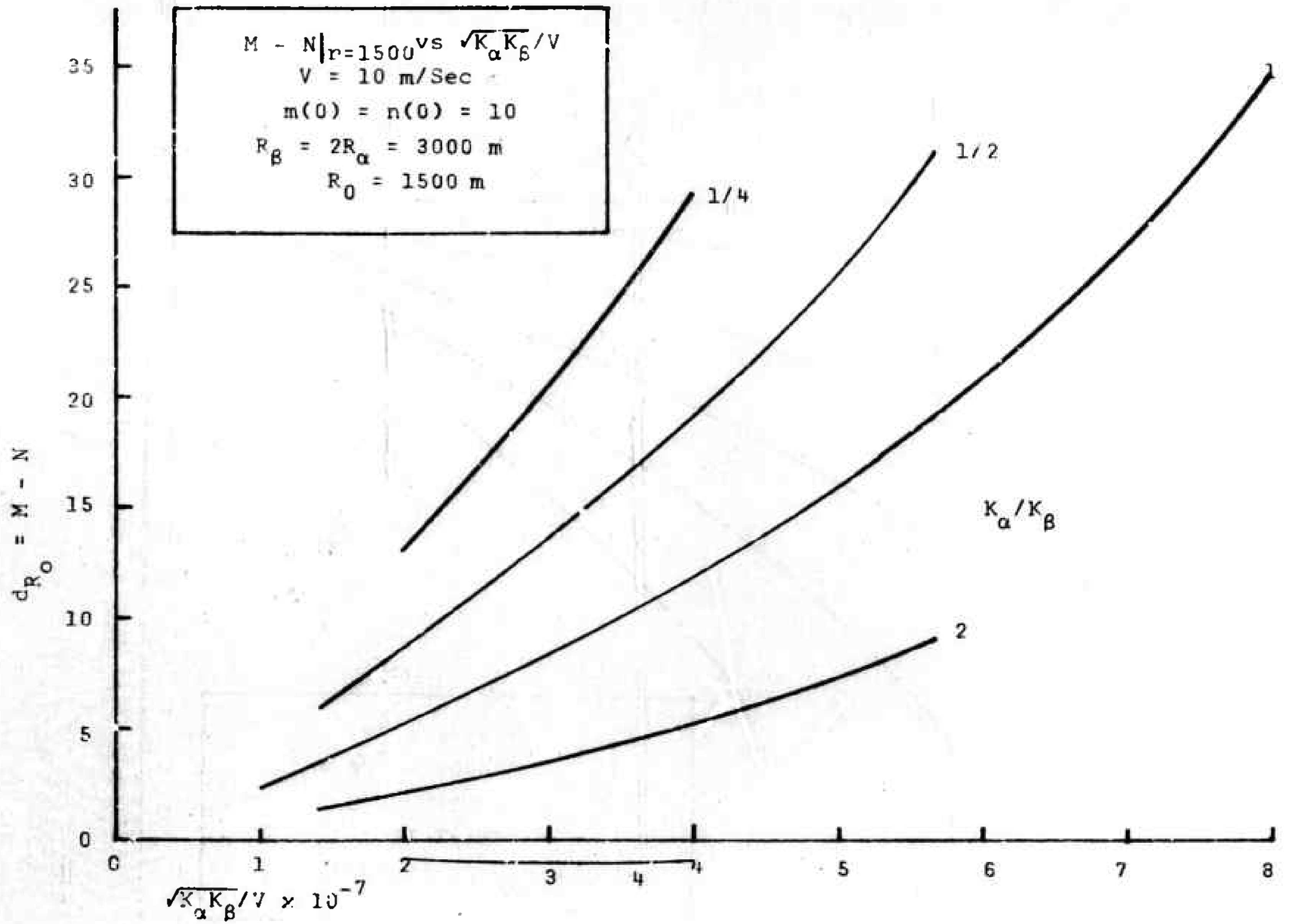


Figure 2.2.7.8B

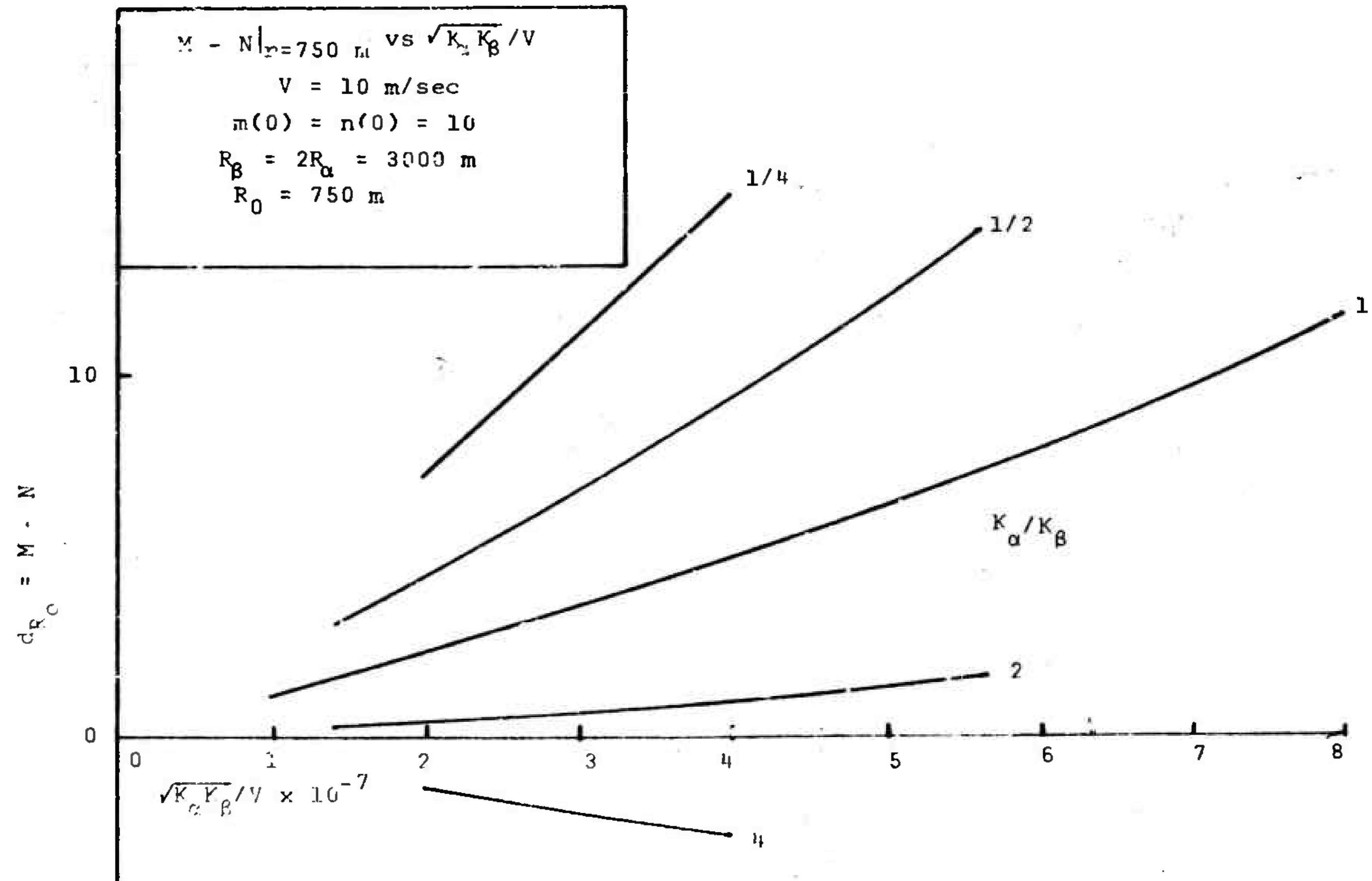
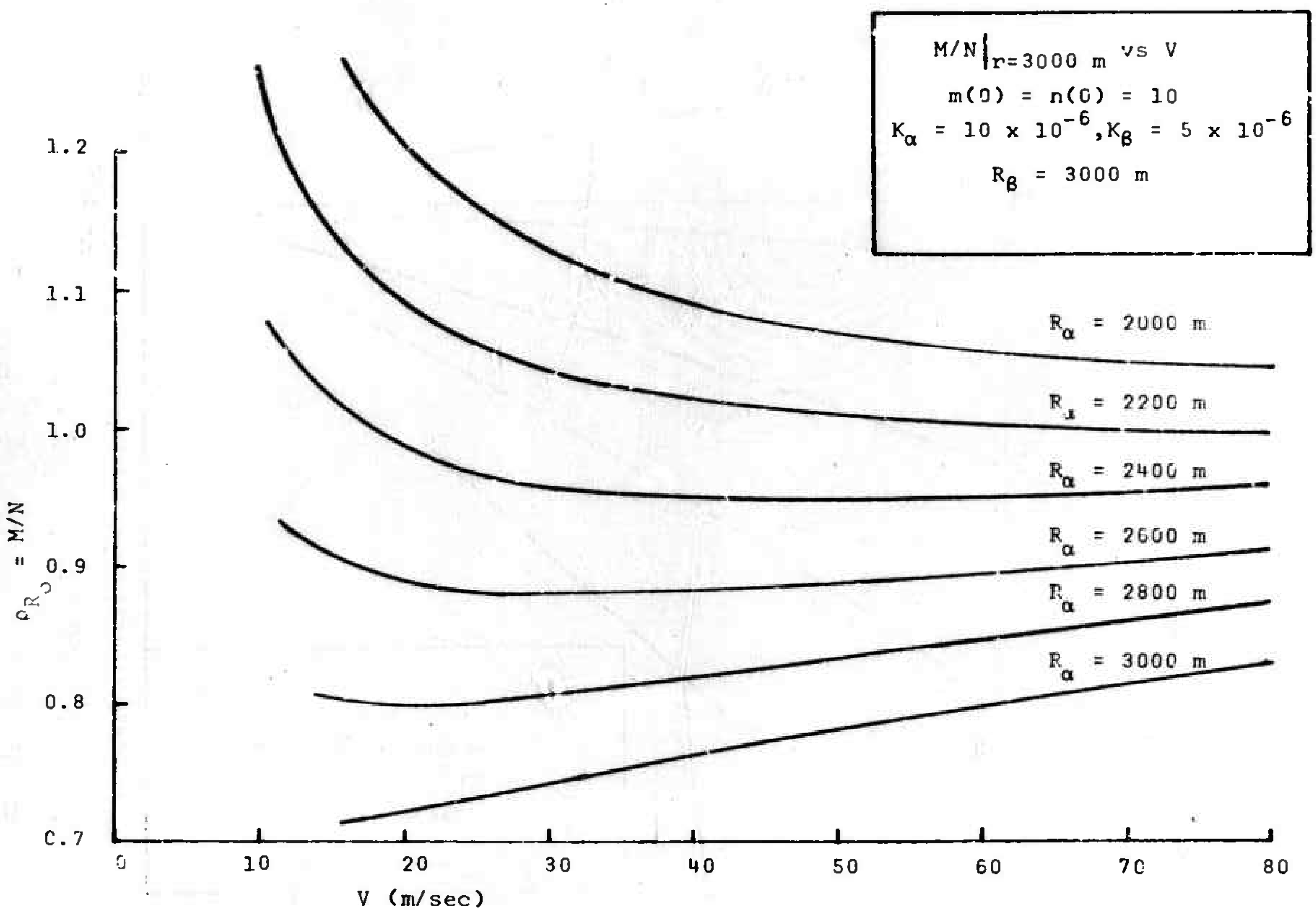
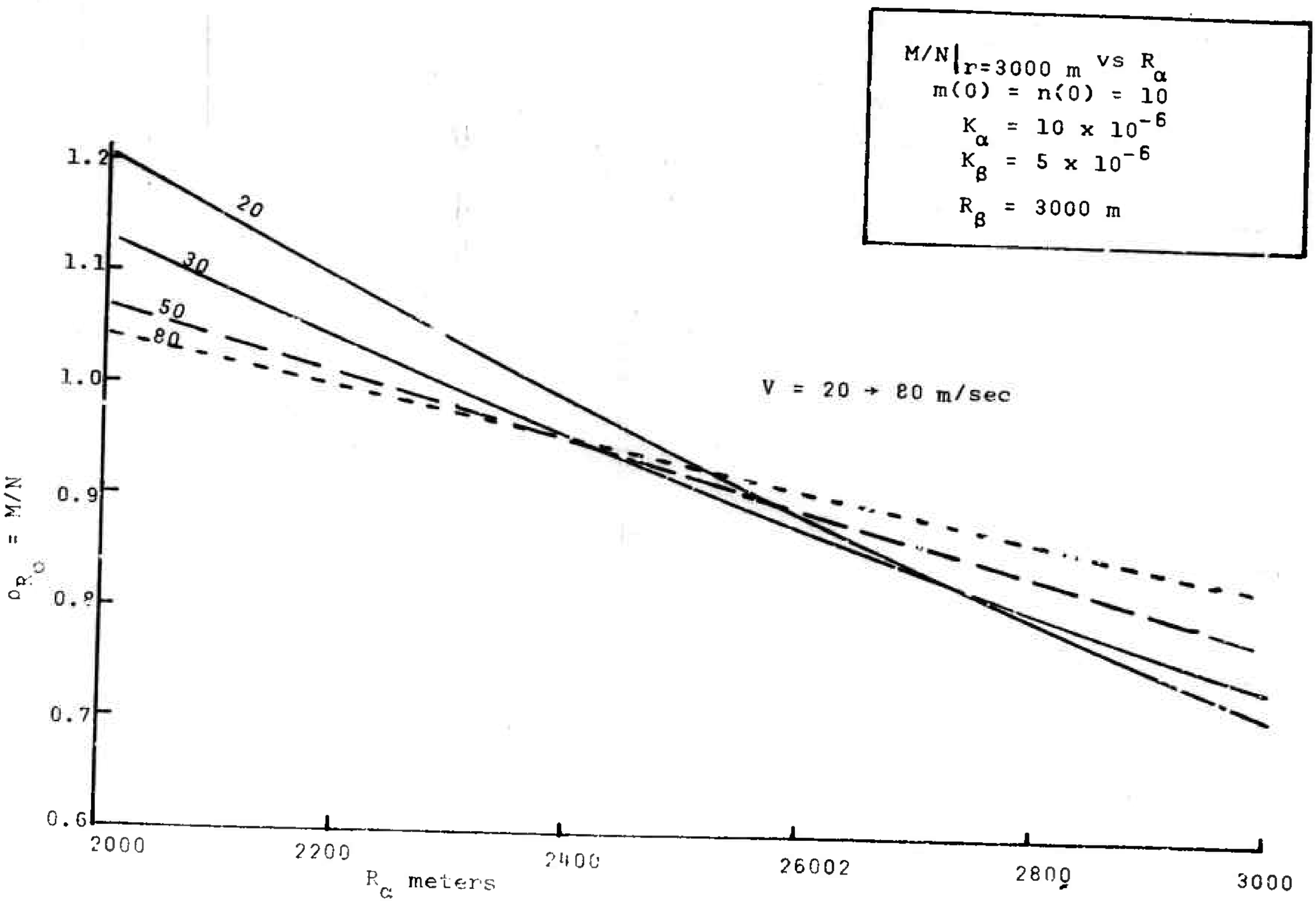


Figure 2.2.7.8C



M/N|r=3000 m vs V
m(0) = n(0) = 10
Kα = 10 x 10^-6, Kβ = 5 x 10^-6
Rβ = 3000 m
Rα = 2000 m
Rα = 2200 m
Rα = 2400 m
Rα = 2600 m
Rα = 2800 m
Rα = 3000 m
= M/N
1.2
1.1
1.0
0.9
0.8
0.7
0
10
20
30
40
50
60
70
80
V (m/sec)

Figure 2.3.1.3



$M/N|_{r=3000\ m}$ vs R_α
$m(0) = n(0) = 10$
$K_\alpha = 10 \times 10^{-6}$
$K_\beta = 5 \times 10^{-6}$
$R_\beta = 3000$ m
V = 20 → 80 m/sec
20
30
50
80
1.2
1.1
1.0
0.9
0.8
0.7
0.6
2000
2200
2400
2600
2800
3000
R_α meters
ρ_{R_α} = M/N

Figure 2.3.3.1

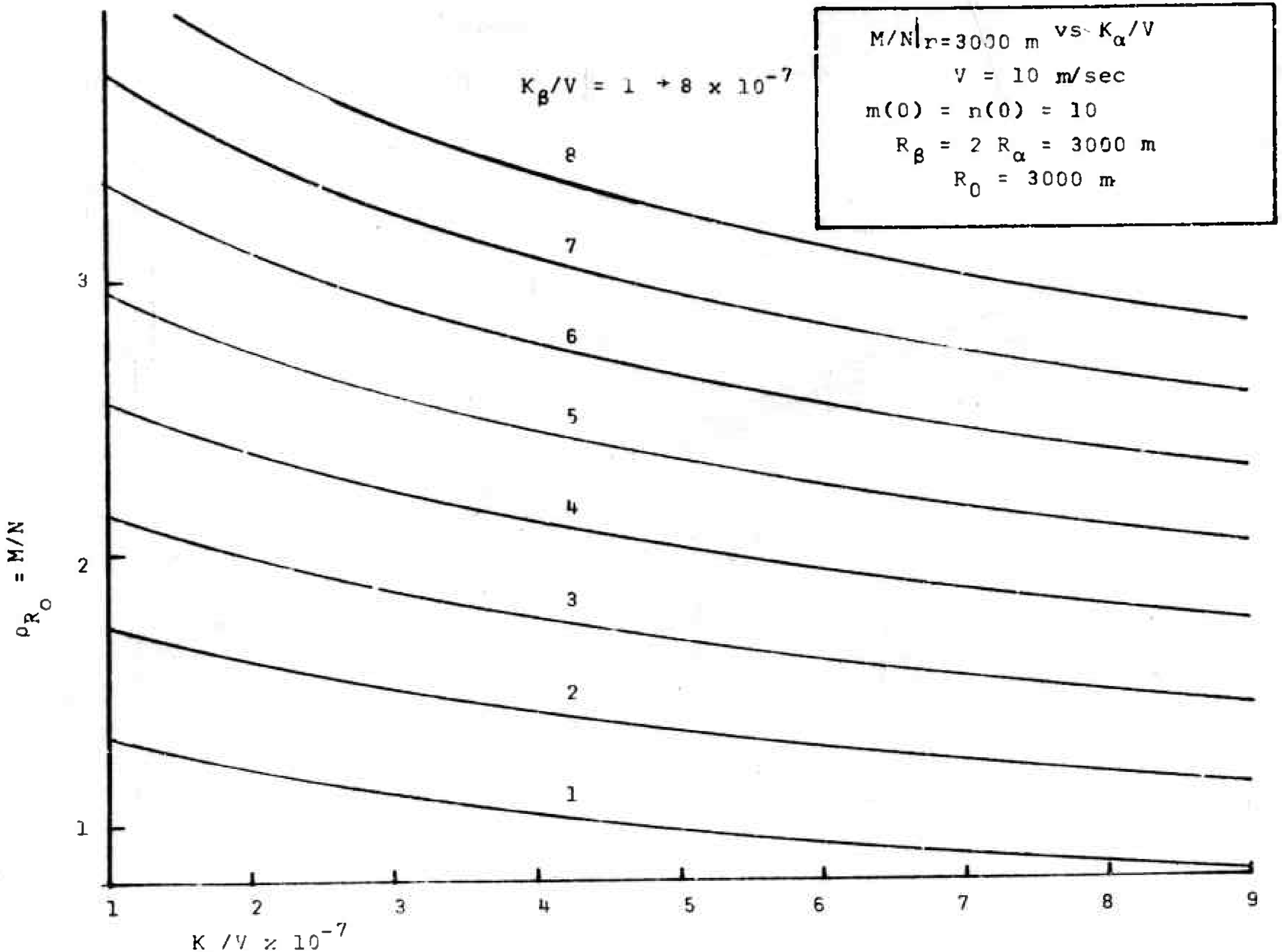


Figure 2.3.5.6A

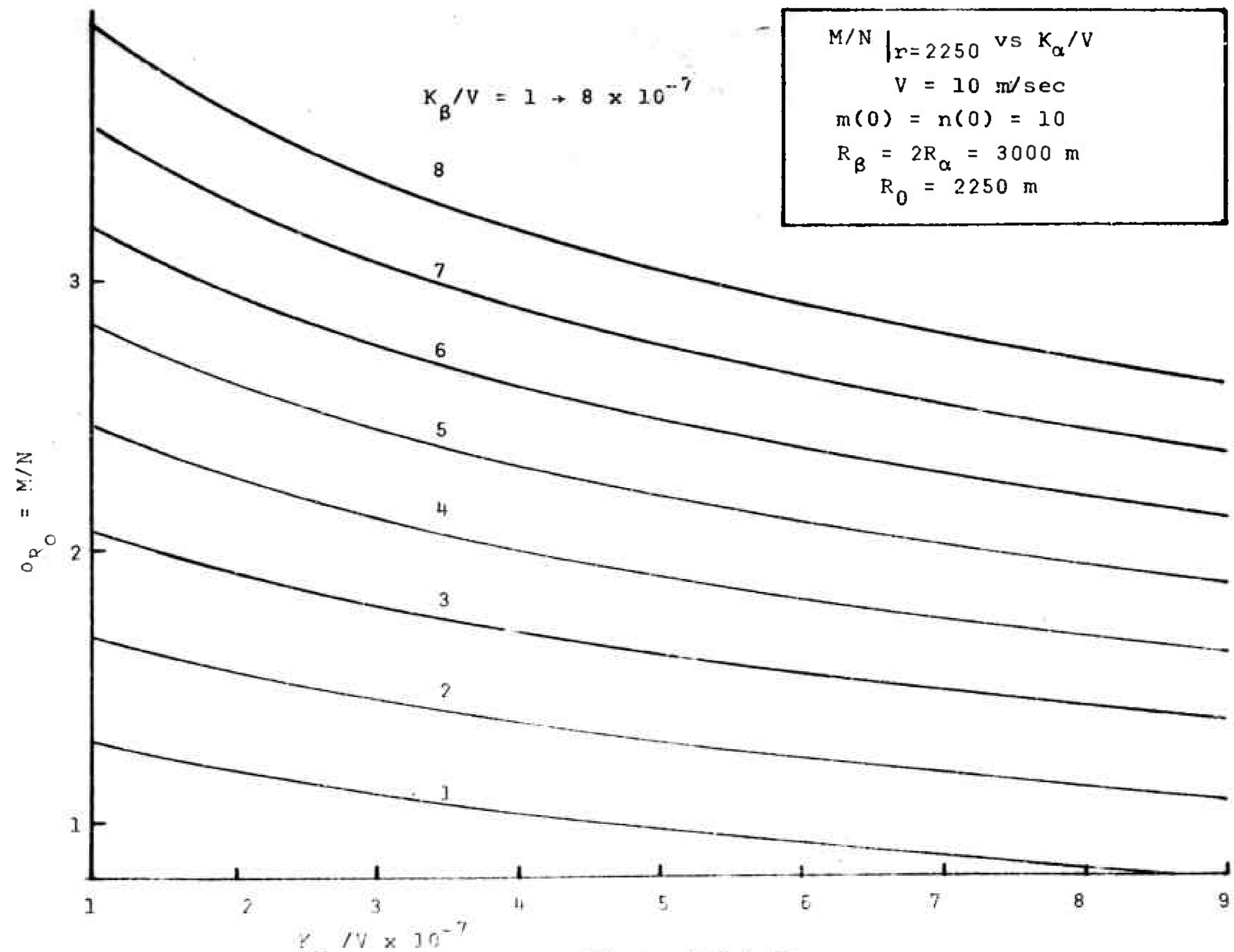


Figure 2.3.5.6B

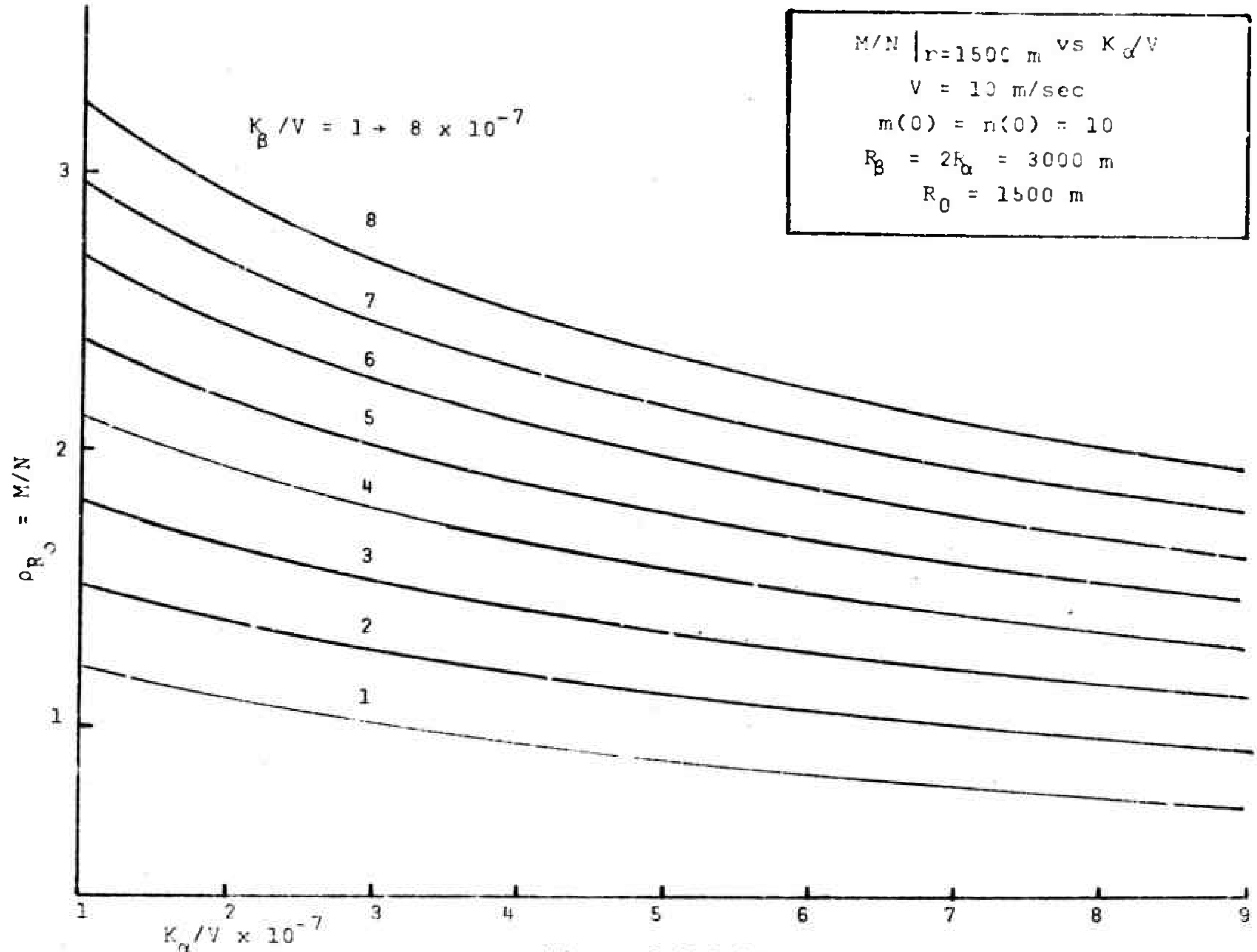


Figure 2.3.5.6C

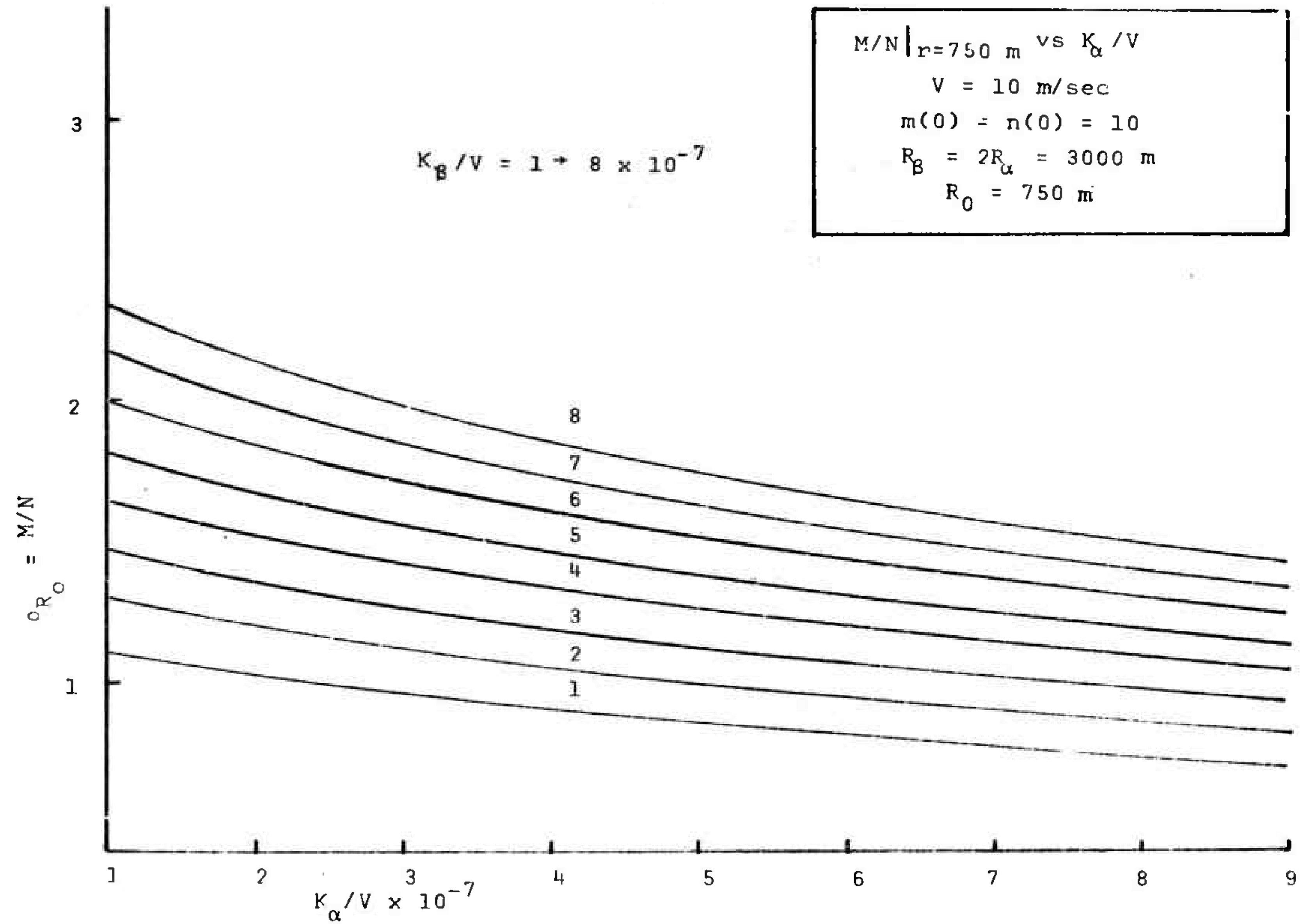


Figure 2.3.5.6D

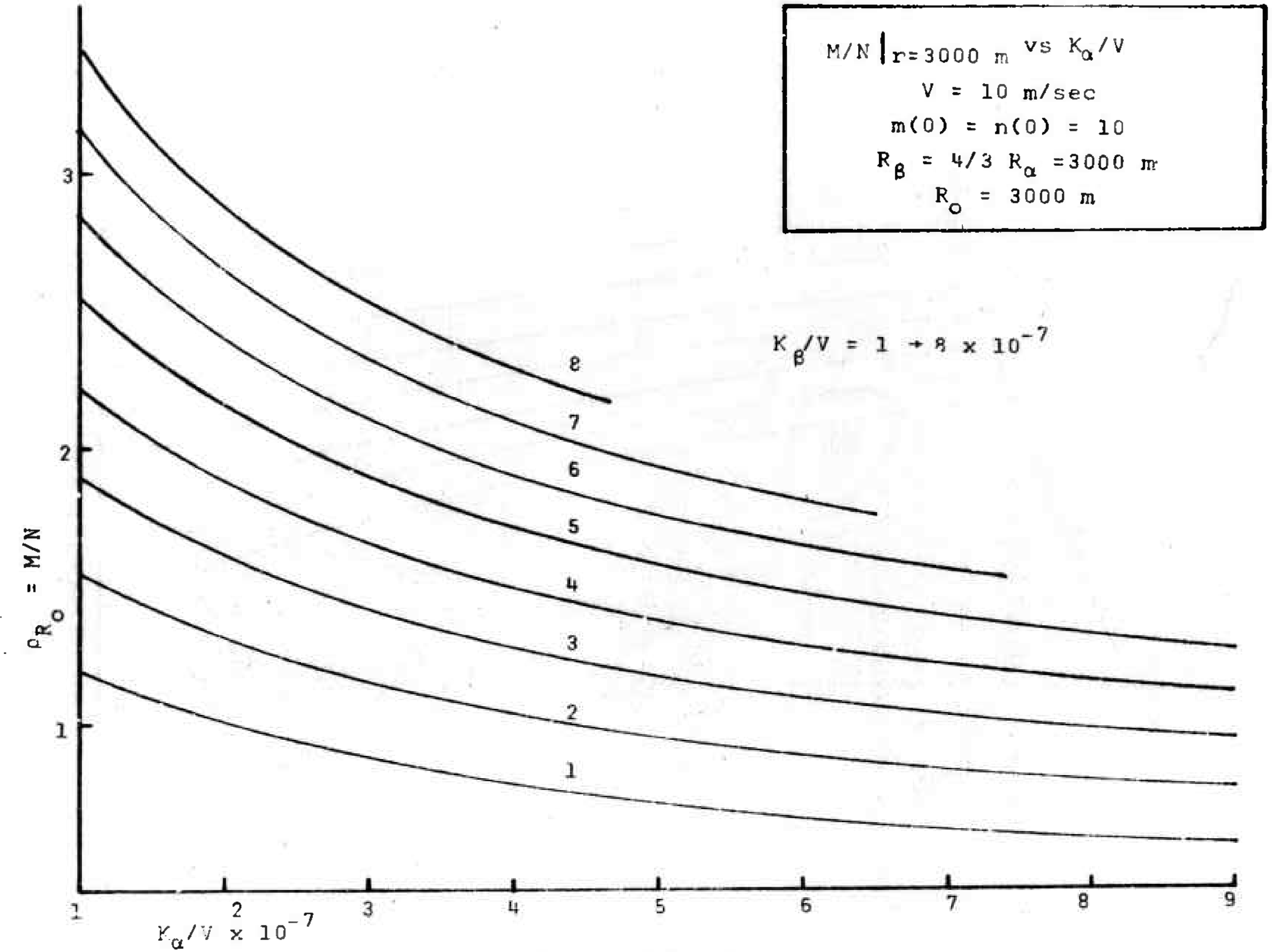


Figure 2.3.5.6E

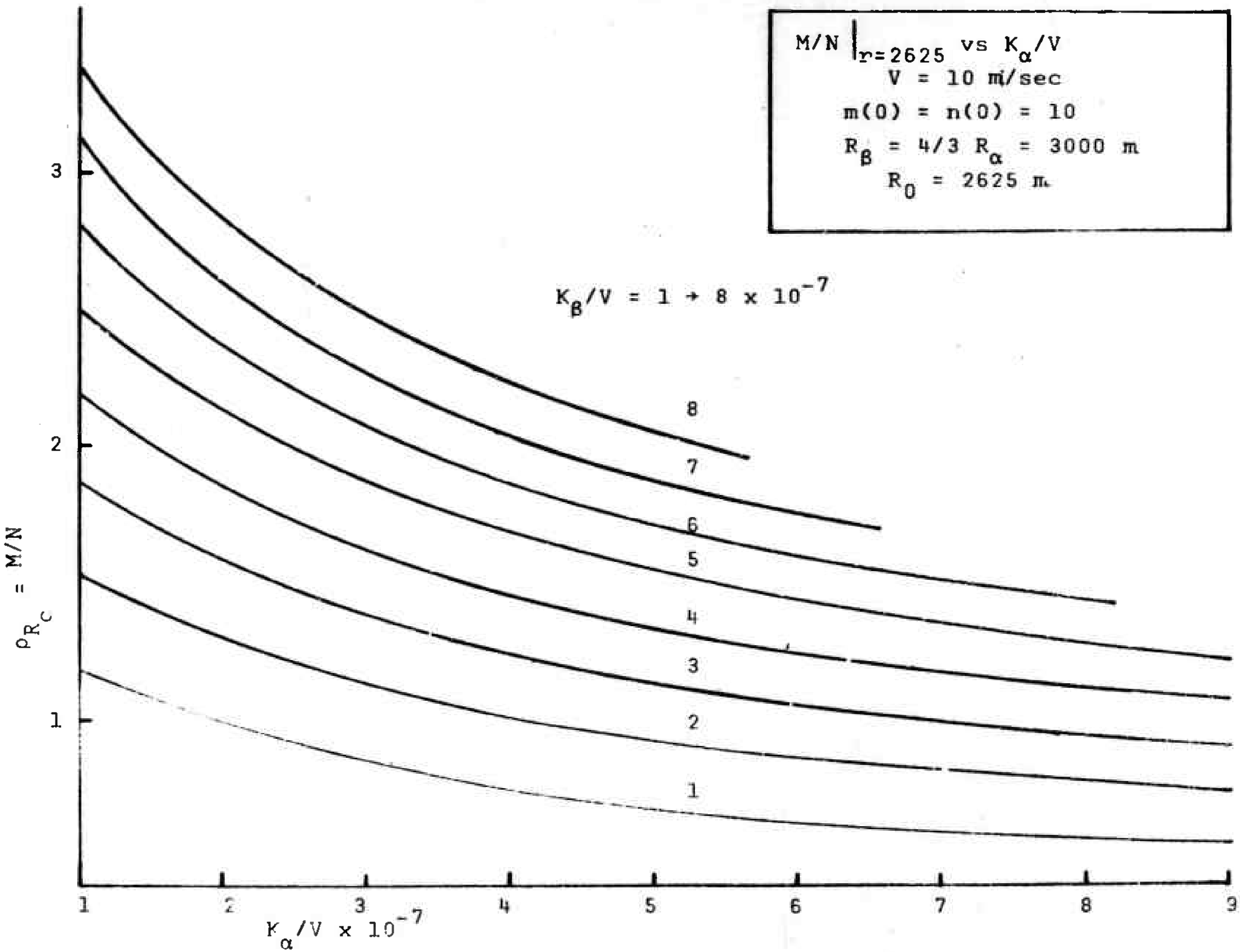


Figure 2.3.5.6F

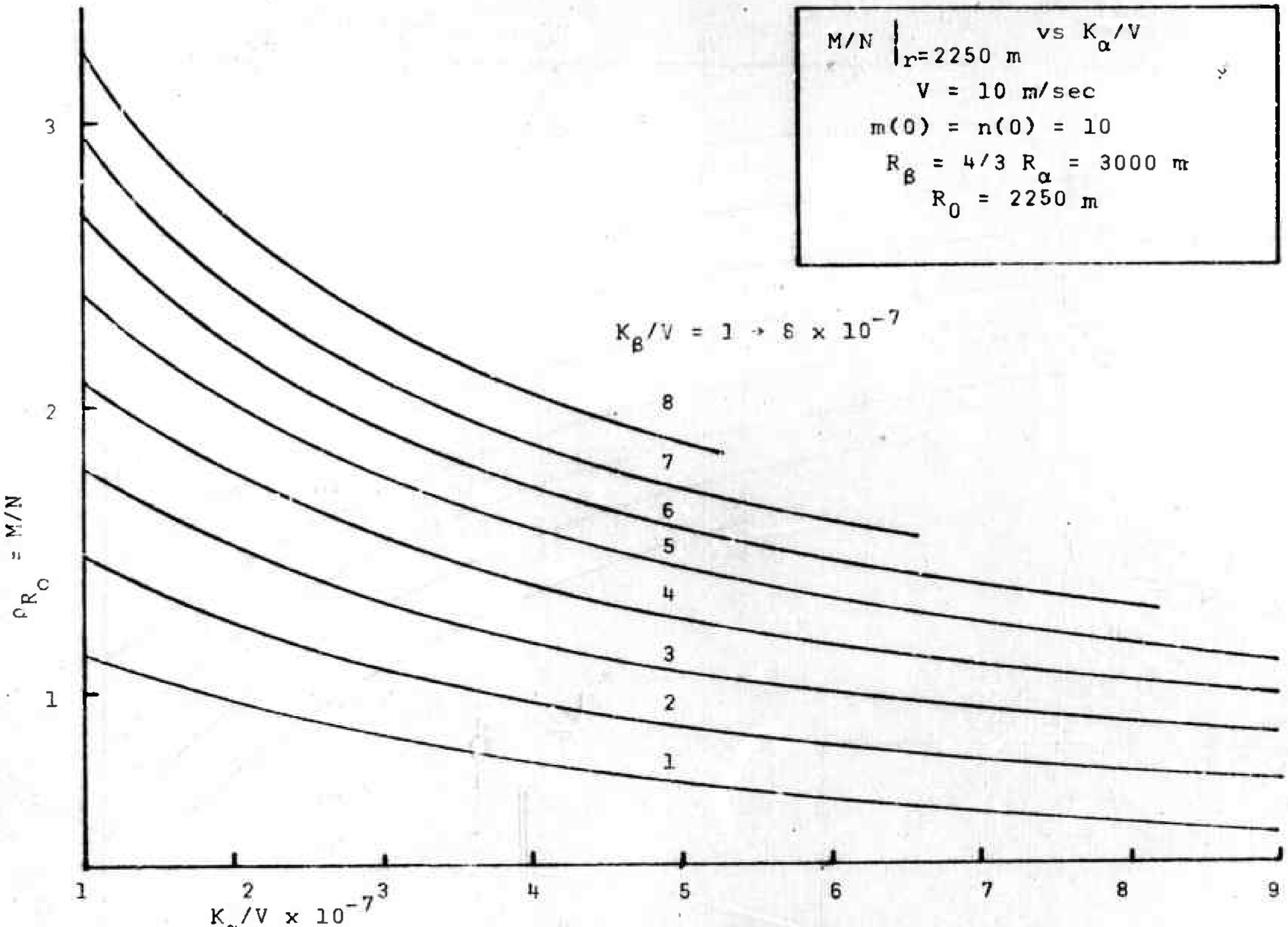


Figure 2.3.5.6G

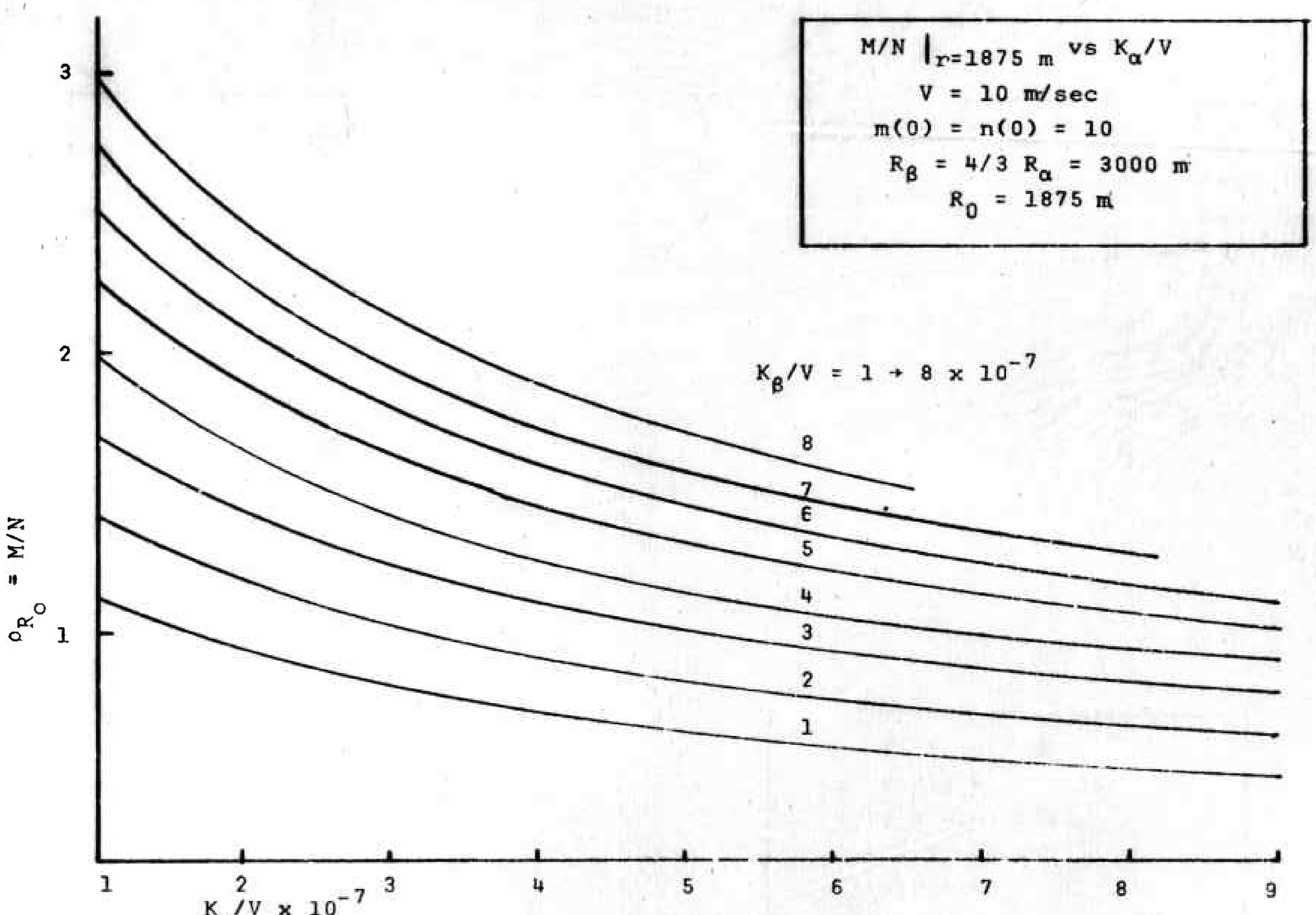


Figure 2.3.5.6H

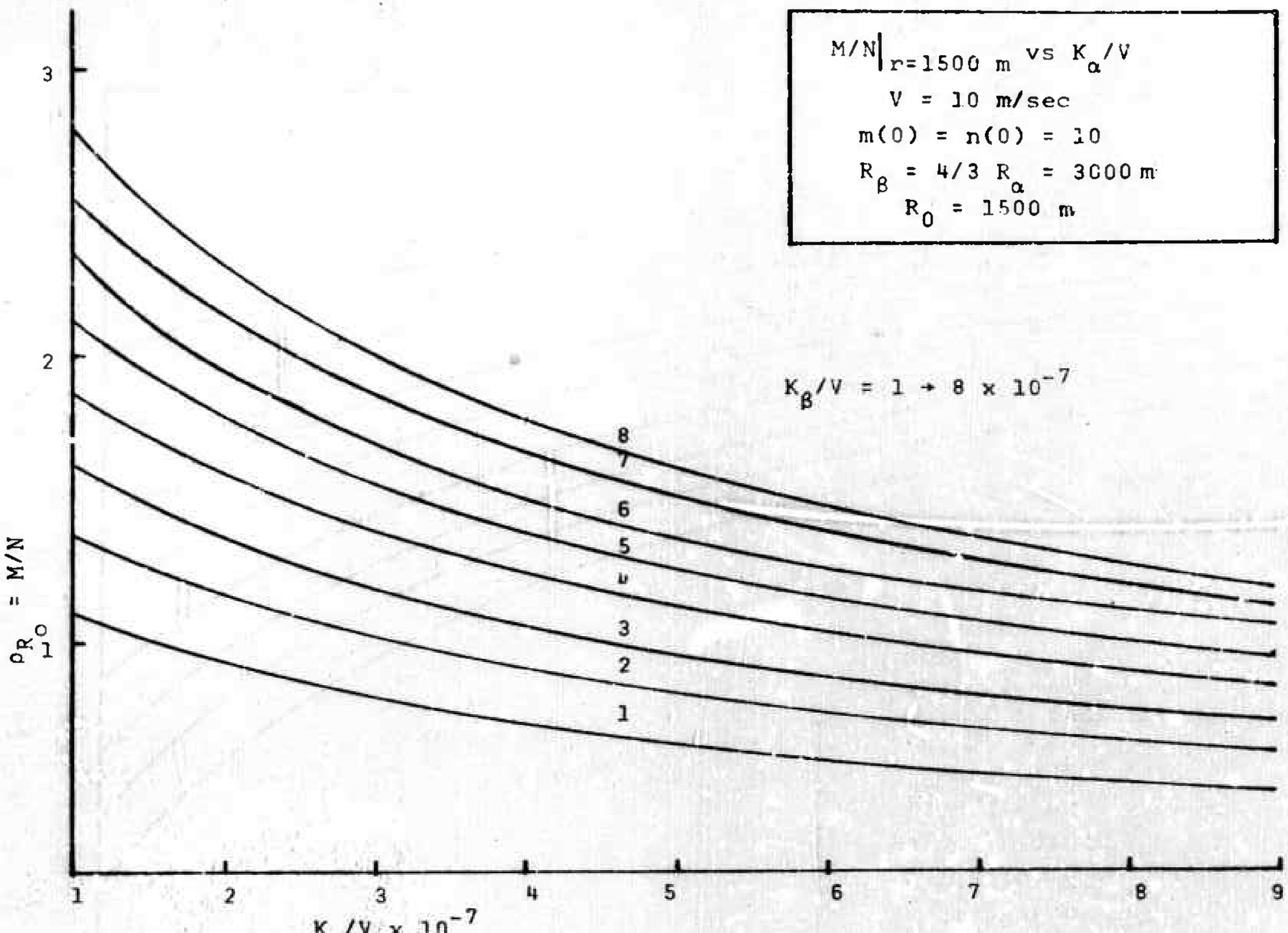


Figure 2.3.5.6I

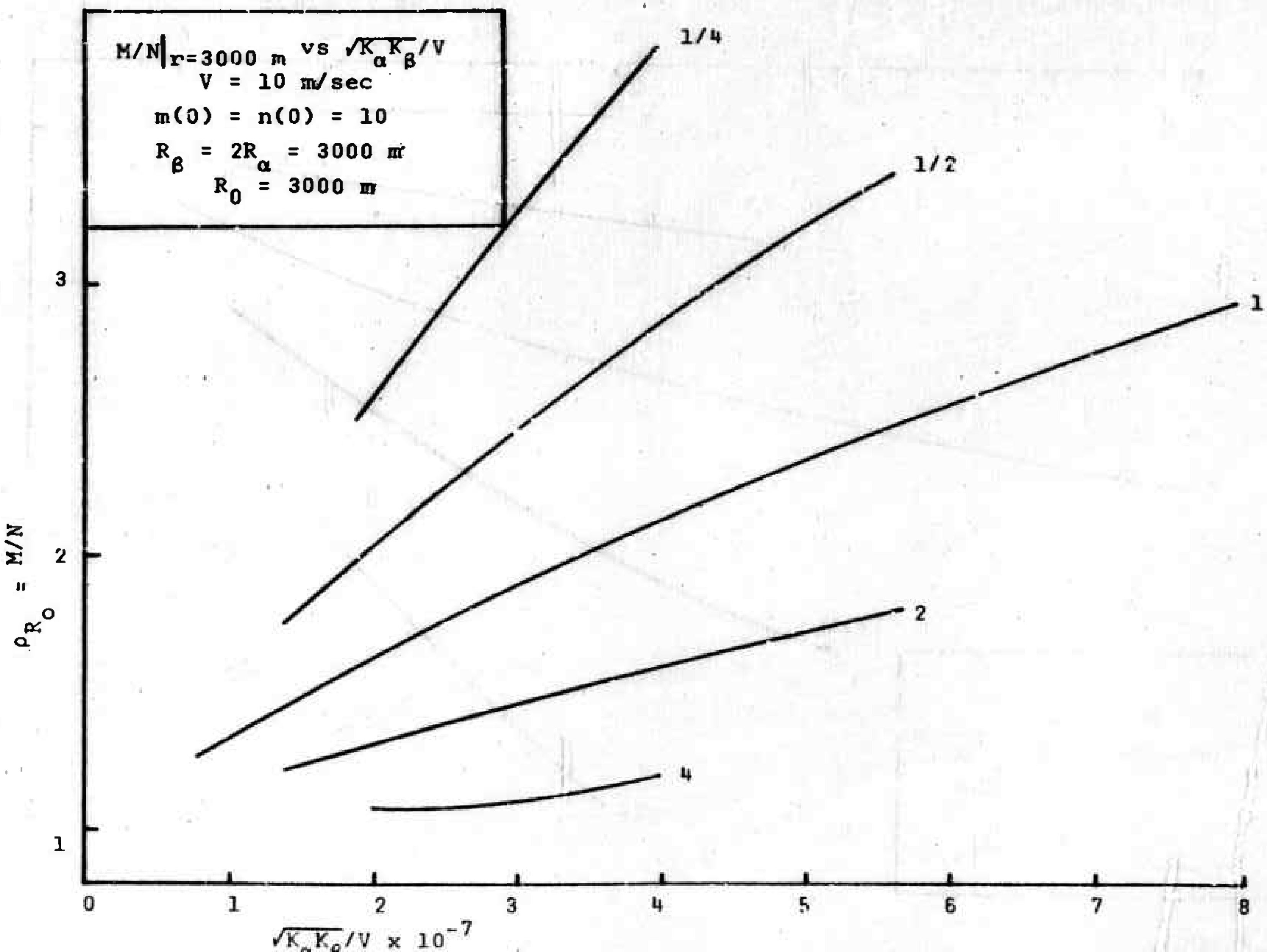


Figure 2.3.7.8A

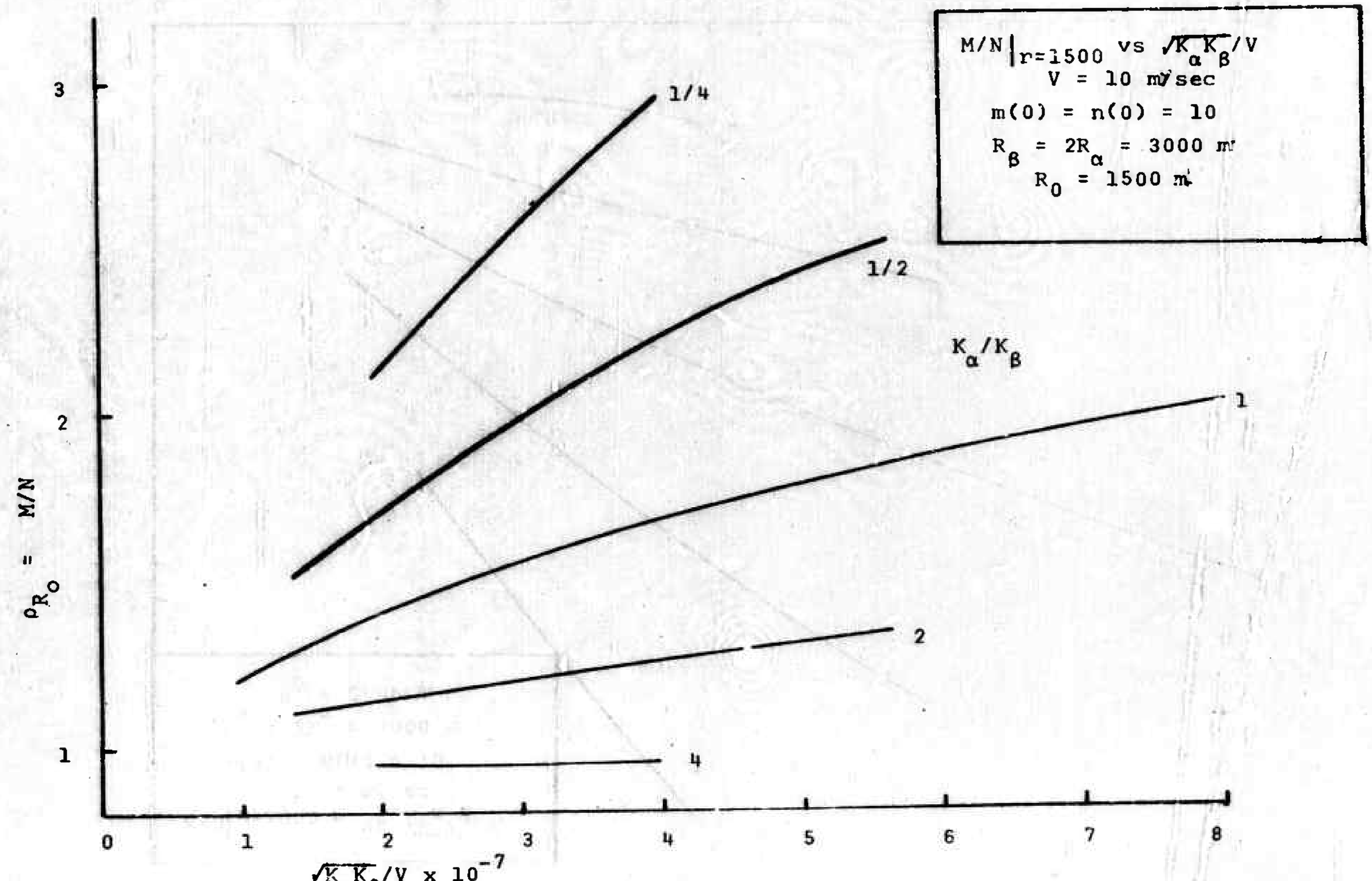


Figure 2.3.7.8B

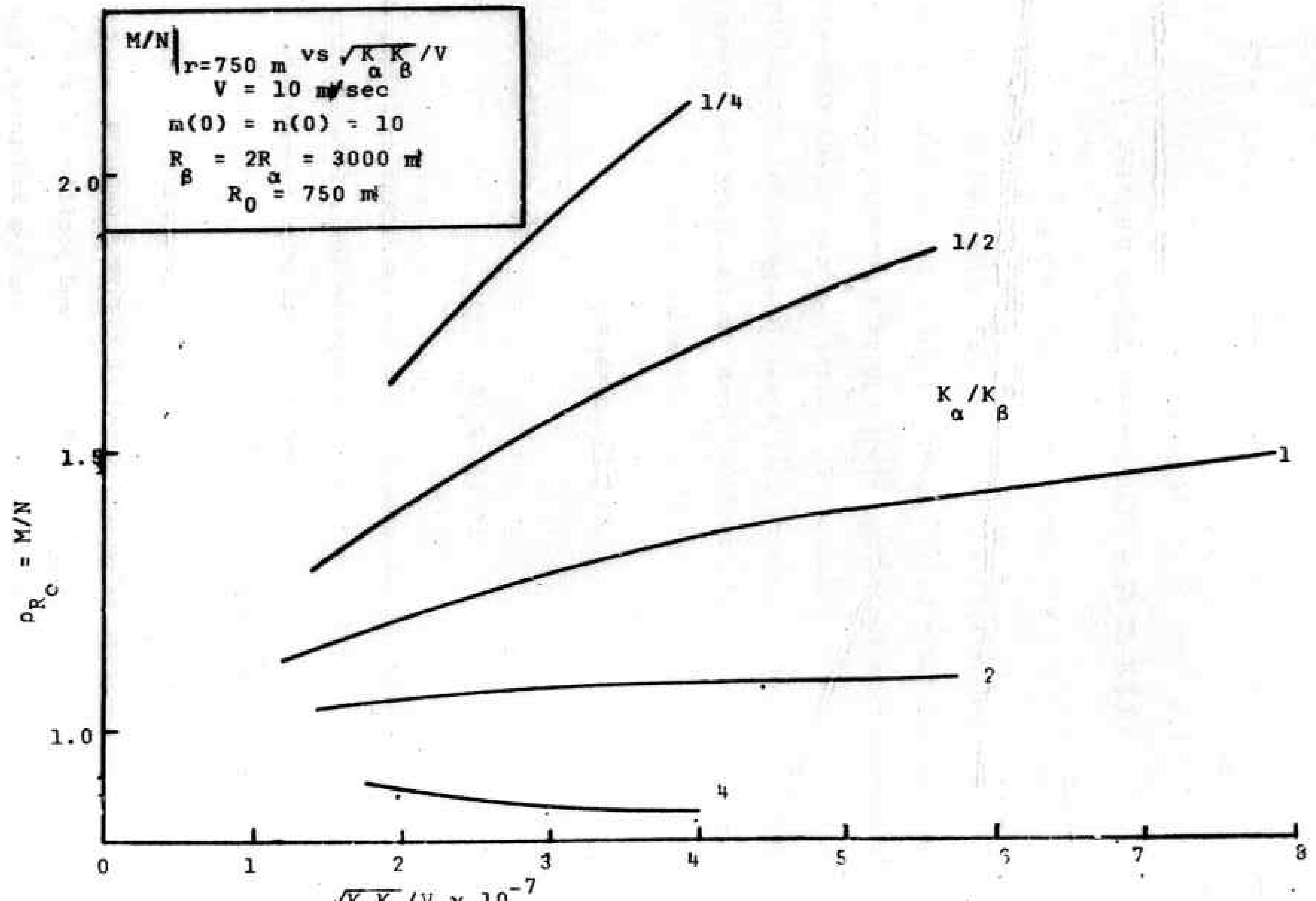


Figure 2.3.7.8C

Chapter 6

DYNAMICS OF A FIRE-SUPPORT ATTACK DOCTRINE

Seth Bonder and George Cooper

Previous chapters of this part of the report considered homogeneous-force battle models in which (a) the ratio of the attrition rate functions $\left(\frac{\beta(r)}{\alpha(r)}\right)$ equals a constant and (b) the ratio was not a constant. In the former case closed form solutions were developed; however, only analytic approximations and analog computer results were obtained when the ratio was not constant. In this chapter we consider the situation in which

$$\alpha(r) = \text{constant} \tag{1}$$

and

$$\beta(r) = K_\alpha(R_\alpha - r) \tag{2}$$

such that $\frac{\beta(r)}{\alpha(r)}$ is not constant but the resultant equations do yield to an analytic solution. A hypothetical, fire-support attack doctrine which possesses this property is described in the following section.

6.1 Tactical Situation

The tactical situation shown in Figure 1 depicts

1. A Red force (n) defending a fixed position at $r = 0$.
2. A Blue force (m), under fire from the Red force, moving from $r = R_o$ (range at which the battle begins) to

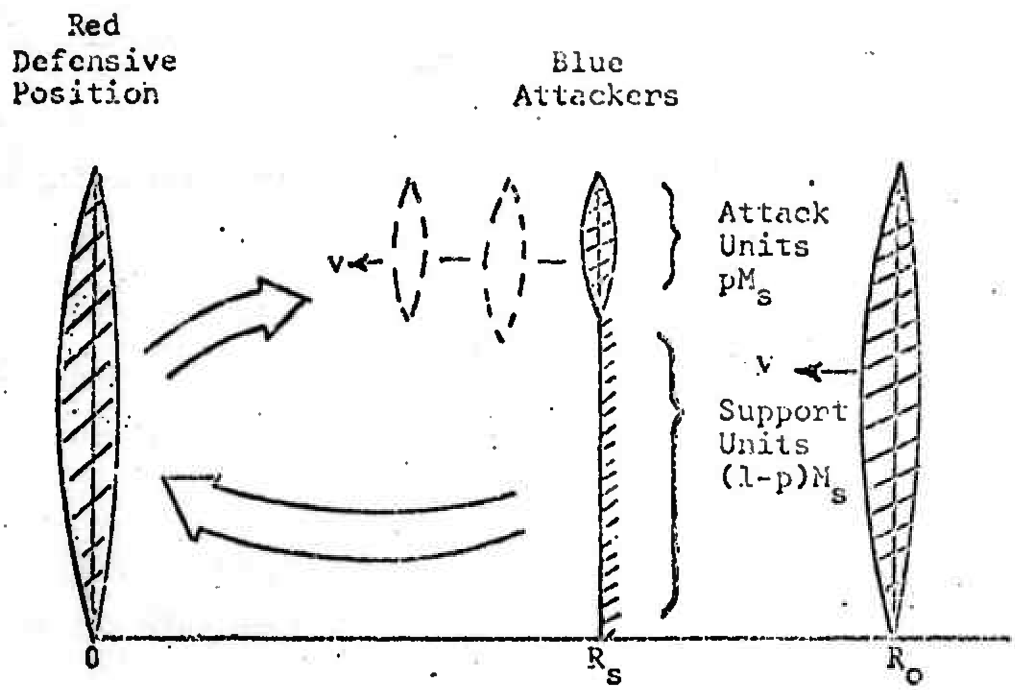


Figure 1 Fire-Support Tactical Situation

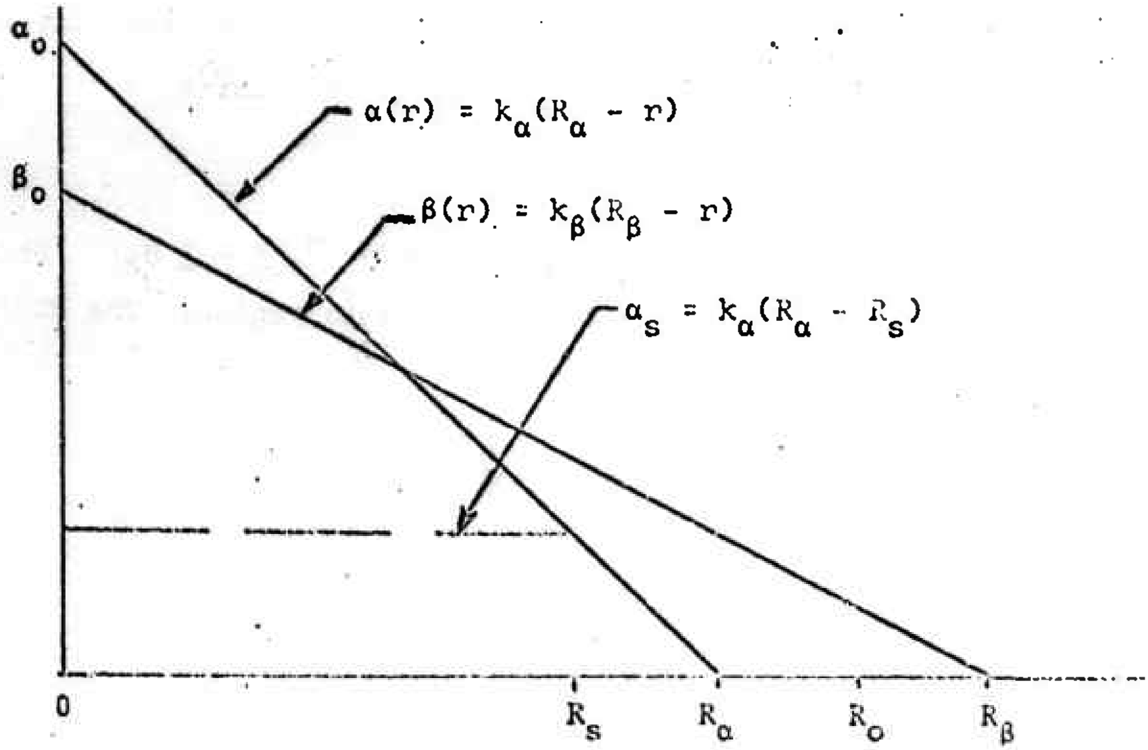


Figure 2 Attrition-Rate Functions for the Fire-Support Situation

$r = R_s$ at a constant speed (v) without returning fire on the Red force.

3. At $r = R_s$, p percent of the remaining Blue force (M_s) continues to advance at speed v without firing. The remaining $(1 - p)M_s$ Blue units stop and provide supporting fire on the Red force.

4. Red fires only on the moving Blue units.

The attrition-rate functions which result from this situation for use in the differential model of combat are shown in Figure 2. The Red force attrition rate varies with range since Red units engage closing Blue units. The Blue attrition rate is a constant, $\alpha_s = K_\alpha(R_\alpha - R_s)$, since the supporting fire Blue units remain a fixed distance, R_s, from the Red units.

6.2 Solution Procedure

Consider first the range interval $R_s \leq r \leq R_o$. The Red forces do not suffer any losses in this region. The Blue loss rate is

$$\frac{dm}{dt} = v\,\frac{dm}{dr} = -\beta(r)N \tag{3}$$

since

$$\beta(r) = K_\beta(R_\beta - r)$$

$$\frac{dm}{dr} = -\frac{K_\beta}{v}(R_\beta - r)N\ . \tag{4}$$

Letting $u = R_\beta - r$, $du = -dr$,

$$dm = \frac{K_\beta}{v} Nu\, du \tag{5}$$

and

$$m = \frac{K_\beta N u^2}{2v} + C$$

$$= \frac{K_\beta N(R_\beta - r)^2}{2v} + C \ .$$

At $r = R_o$, $n = N$ and $m = M$; therefore,

$$C = M - \frac{K_\beta N}{2v} (R_\beta - R_o)^2$$

and

$$m = M + \frac{K_\beta N}{2v} [(R_\beta - r)^2 - (R_\beta - R_o)^2] \ . \tag{6}$$

At range $r = R_s$,

$$m = M_s = M + \frac{K_\beta N}{2v} [(R_\beta - R_s)^2 - (R_\beta - R_o)^2] \tag{7}$$

$$R_s \leq r \leq R_o \ .$$

Consider next the range interval $0 \leq r \leq R_s$. Let

$m_1 = pM_s$ = number in the Blue moving force

$M_2 = (1 - p)M_s$ = number in the Blue fire-support force.

Then,

$$\frac{dM_2}{dt} = 0$$

since the fire-support force is not fired upon.

The Red force loss rate

$$\frac{dn}{dt} = v\,\frac{dn}{dr} = -\alpha_s M_2 , \tag{8}$$

where

$$\alpha_s = K_\alpha (R_\alpha - R_s)$$

is the Blue force attrition rate at $r = R_s$. From (8)

$$n = -\frac{\alpha_s M_2}{v}\,r + C , \tag{9}$$

and since $n = N$ at $r = R_s$,

$$C = N + \frac{\alpha_s M_2}{v}\,R_s .$$

Thus

$$n = N + \frac{\alpha_s M_2}{v}\,(R_s - r) . \tag{10}$$

The moving Blue force loss rate

$$\frac{dm_1}{dt} = v\,\frac{dm_1}{dr} = -\beta(r)n$$

$$= -K_\beta (R_\beta - r)n . \tag{11}$$

Substituting (9) into (11),

$$\frac{dm_1}{dr} = \frac{-K_\beta(R_\beta - r)}{v} \left[N + \frac{\alpha_s M_2}{v} (R_s - r) \right]$$

$$= \frac{-K_\beta N(R_\beta - r)}{v} - K[R_s(R_\beta - r) - R_\beta r + r^2], \quad (12)$$

where, if v is not a function of range (i.e., a constant speed assault), the constant

$$K = \frac{\alpha_s M_2 K_\beta}{v^2} .$$

Integrating (12),

$$m_1 = \frac{K_\beta N(R_\beta - r)^2}{2v} + K\left[+ \frac{R_s(R_\beta - r)^2}{2} + \frac{R_\beta r^2}{2} - \frac{r^3}{3} \right] + C .$$

Employing the initial condition that at $r = R_s$, $m_1 = M_1 = pM_s$,

$$C = M_1 - \frac{K_\beta N(R_\beta - R_s)^2}{2v} - \frac{KR_s(R_\beta - R_s)^2}{2} - \frac{KR_\beta R_s{}^2}{2} + \frac{KR_s{}^3}{3}$$

and

$$m_1 = M_1 + \frac{K_\beta N}{2v} [(R_\beta - r)^2 - (R_\beta - R_s)^2] + \frac{KR_s}{2} [(R_\beta - r)^2 - (R_\beta - R_s)^2]$$

$$+ \frac{KR_\beta}{2} \left[R_\beta r^2 - R_\beta R_s{}^2 - \frac{2r^3}{3} + \frac{2R_s{}^3}{3} \right] . \quad (13)$$

Adding and subtracting $\frac{K}{2} (R_\beta - R_s)^3$ and collecting similar terms,

$$m_1 = M_1 + \frac{K}{3}\left[(R_\beta - r)^3 - (R_\beta - R_s)^3\right]$$

$$+ \frac{K_\beta N}{2v}\left[(R_\beta - r)^2 - (R_\beta - R_s)^2\right] \tag{14}$$

$$- \frac{K(R_\beta - R_s)}{2}\left[(R_\beta - r)^2 - (R_\beta - R_s)^2\right]$$

$$0 \le r \le R_s ,$$

where it is remembered that

$$K = \frac{\alpha_s M_2 K_\beta}{v^2} = \frac{K_\alpha (R_\alpha - R_s) M_2 K_\beta}{v^2} . \tag{15}$$

6.3 Conditions on m_1 and r Approaching Zero

We consider next the conditions such that m_1 and n approach zero simultaneously. Let the range at which this occurs be denoted by R^o. Then, one has from (14)

$$m_1 = M_1 + \frac{K}{3}\left[(R_\beta - R^o)^3 - (R_\beta - R_s)^3\right] + \left[\frac{K_\beta N}{2v} - \frac{K(R_\beta - R_s)}{2}\right]$$

$$\cdot \left[(R_\beta - R^o)^2 - (R_\beta - R_s)^2\right]$$

$$= 0 \tag{16}$$

and

$$n = N + \frac{\alpha_s M_2}{v}(R_s - R^o) = 0 . \tag{17}$$

Letting $\theta = \frac{Nv}{\alpha_s M}$, the range R^o can be found from (17) to be

$$R^o = \frac{Nv}{\alpha_s M_2} + R_s = \theta + R_s \ . \tag{18}$$

Letting $\Pi = R_\beta - R_s$, and substituting (18) into (16), one has

$$M_1 + \frac{K}{3}\{[R_\beta - (\theta + R_s)]^3 - \Pi^3\} + \left[\frac{K_\beta N}{2v} - \frac{K\Pi}{2}\right]$$

$$\cdot \{[R_\beta - (\theta + R_s)]^2 - \Pi^2\} = 0$$

$$M_1 + \frac{K}{3}\left[-3\Pi^2\theta + 3\Pi\theta^2 - \theta^3\right] + \left[\frac{K_\beta N}{2v} - \frac{K\Pi}{2}\right]\left[-2\Pi\theta + \theta^2\right] = 0 \ .$$

Since $K = \frac{\alpha_s M_2 K_\beta}{v^2}$,

$$M_1 - \frac{\alpha_s M_2 K_\beta}{3v^2}\left(\frac{Nv}{\alpha_s M_2}\right)^3 + \frac{\alpha_s M_2 K_\beta \Pi}{2v^2}\left(\frac{Nv}{\alpha_s M_2}\right)^2 + \frac{K_\beta N}{2v}\left(\frac{Nv}{\alpha_s M_2}\right)^2$$

$$- \frac{\Pi K_\beta N}{v}\left(\frac{Nv}{\alpha_s M_2}\right) = 0 \ .$$

Finally,

$$M_1 - \frac{K_\beta \Pi N^2}{2\alpha_s M_2} + \frac{K_\beta N^3 v}{6\alpha_s^2 M_2^2} = 0 \ . \tag{19}$$

Remembering that

$$M_1 = pM_s$$

$$M_2 = qM_s \quad \text{where } q = 1 - p$$

$$M_s = M + \frac{K_\beta N}{2v} [(R_\beta - R_s)^2 - (R_\beta - R_o)^2]$$

and letting

$$\gamma = (R_\beta - R_s)^2 - (R_\beta - R_o)^2 ,$$

(19) becomes

$$pq^2 \frac{M_s^3}{N^3} - \frac{K_\beta \Pi q}{2\alpha} \frac{M_s}{N} + \frac{K_\beta v}{6\alpha_s^2} = 0 \tag{20}$$

or

$$z^3 - \frac{K_\beta \Pi z}{2pq\alpha_s} + \frac{K_\beta v}{6pq^2\alpha_s^2} = 0 , \tag{21}$$

where $z = \frac{M_s}{N} = \frac{M}{N} + \frac{K_\beta \gamma}{2v}$. To find the desired conditions, it is necessary to solve for the three roots of (21) denoted by r_1, r_2, and r_3. This is accomplished by applying Cordan's formulas.

Equation 21 is in the general form $x^3 + bx^2 + cx + d = 0$, where $b = 0$, $c = - \frac{K_\beta \Pi}{2pq\alpha_s}$, and $d = \frac{K_\beta v}{6pq^2\alpha_s^2}$. Intermediate quantities needed are s, t, and L, where:

$$s = c - \frac{1}{3}b^2 = - \frac{K_\beta \Pi}{2pq\alpha_s} \tag{22}$$

$$t = d - \frac{1}{3}bc + \frac{2}{27}b^3 = \frac{K_\beta v}{6pq^2\alpha_s^2} \tag{23}$$

$$L = \frac{1}{27}s^3 + \frac{1}{4}t^2 = -\frac{K_\beta^3 \Pi^3}{216p^3q^3\alpha_s^3} + \frac{K_\beta^2 v^2}{144p^2q^4\alpha_s^4} \; . \tag{24}$$

It can be shown that equation 21 has three distinct real roots, a single real root, or at least two equal real roots according to whether L is negative, positive, or zero, respectively. The latter condition will be neglected for the moment. The condition that L be positive implies that

$$\beta_s\alpha_s < \frac{\frac{3}{2}pv^2}{(R_\beta - R_s)^2(1 - p)} \; . \tag{25}$$

The single real root is given by

$$r_1 = \sqrt[3]{A} + \sqrt[3]{B} \; ,$$

where

$$A = -\frac{1}{2}s + \sqrt{L}$$

$$B = -\frac{1}{2}s - \sqrt{L}$$

$$\sqrt[3]{A} = \left[-\frac{1}{2}\,\frac{K_\beta v}{6pq^2\alpha_s^2} + \frac{K_\beta}{12pq^2\alpha_s^2}\sqrt{\frac{3pv^2 - 2K_\beta\Pi^3 q\alpha_s}{3p}}\right]^{1/3}$$

$$\sqrt[3]{B} = \left[-\frac{K_\beta}{12pq^2\alpha_s^2}\right]^{1/3}\left[\sqrt{\frac{3pv^2 - 2K_\beta\Pi^3 q\alpha_s}{3p}} + v\right]^{1/3}$$

and

$$r_1 = \left[\frac{K_\beta}{12\sqrt{3p}\; pq^2\alpha_s^2}\right]^{1/3} \left[\left(\sqrt{3pv^2 - 2K_\beta\Pi^3\alpha_s q} - \sqrt{3pv^2}\right)^{1/3}\right.$$

$$\left. - \left(\sqrt{3pv^2 - 2K_\beta\Pi^3\alpha_s q} + \sqrt{3pv^2}\right)^{1/3}\right] . \qquad (26)$$

When $L < 0$, three real roots exist. These are found by a different procedure, which employs the following quantities:

$$r = \sqrt{-\frac{1}{27}s^3} = \sqrt{\frac{1}{27}\left(\frac{K_\beta}{2pq\alpha_s}\right)^3}$$

$$\cos\theta = \frac{-\frac{1}{2}s}{r} = -\frac{v}{\Pi}\sqrt{\frac{3}{2}\frac{p}{qK_\beta\Pi\alpha_s}} \qquad (27)$$

$$r^{1/3} = \left[\left(\frac{1}{27}\right)^{1/3}\left(\frac{K_\beta\Pi}{2pq\alpha_s}\right)\right]^{1/2} = \sqrt{\frac{K_\beta\Pi}{6pq\alpha_s}} . \qquad (28)$$

Using (27) and (28) in the following expressions:

$$r_1 = 2\sqrt[3]{r}\cos\left(\frac{1}{3}\theta\right)$$

$$r_2 = 2\sqrt[3]{r}\cos\left[\frac{1}{3}(\theta + 360^\circ)\right]$$

$$r_3 = 2\sqrt[3]{r}\cos\left[\frac{1}{3}(\theta + 720^\circ)\right] ,$$

the three roots are

$$r_1 = \sqrt{\frac{2}{3}\frac{K_\beta\Pi}{pq\alpha_s}}\cos\left\{\frac{1}{3}\cos^{-1}\left(-\frac{v}{\Pi}\sqrt{\frac{3}{2}\frac{p}{qK_\beta\Pi\alpha_s}}\right)\right\} \qquad (29)$$

$$r_2 = \sqrt{\frac{2}{3}\frac{K_\beta\Pi}{pq\alpha_s}}\cos\left\{\frac{1}{3}\cos^{-1}\left(-\frac{v}{\Pi}\sqrt{\frac{3}{2}\frac{p}{qK_\beta\Pi\alpha_s}} + 360^\circ\right)\right\} \qquad (30)$$

$$r_3 = \sqrt{\frac{2}{3}\frac{K_\beta\Pi}{pq\alpha_s}}\cos\left\{\frac{1}{3}\cos^{-1}\left(-\frac{v}{\Pi}\sqrt{\frac{3}{2}\frac{p}{qK_\beta\Pi\alpha_s}} + 720^\circ\right)\right\} \ . \qquad (31)$$

Using (26), (29), (30), or (31) as appropriate, the conditions which cause m_1 and n to approach zero simultaneously are easily found.

8.4 Condition for $(m_1 - n) \geq 0$ at the Defended Position

The value of the quantity $d_o = (m_1 - n)$ at zero range is a measure of future success in taking the defended position. Consider the conditions under which $d_o \geq 0$ at range zero. Using previous notation,

$$m_1 = M_1 + \frac{K}{3}\left(R_\beta^3 - \Pi^3\right) + \left(\frac{K_\beta N}{2v} - \frac{K\Pi}{2}\right)\left(R_\beta^2 - \Pi^2\right) \qquad (32)$$

$$= n = N + \frac{\alpha_s M_2 R_s}{v} \ .$$

$d_o \geq 0$ implies that

$$\left(\frac{M}{N} + \frac{K_\beta \gamma}{2v}\right)\left[p + \frac{\alpha_s q K_\beta}{3v^2}(R_\beta^3 - \Pi^3) - \frac{\alpha_s q K_\beta \Pi}{2v^2}(R_\beta^2 - \Pi^2) - \frac{\alpha_s q R_s}{v}\right]$$
$$+ \left[\frac{K_\beta (R_\beta^2 - \Pi^2)}{2v}\right] - 1 \geq 0 \,, \tag{33}$$

or that the initial force ratio must be

$$\frac{M}{N} \geq \frac{1 - \dfrac{K_\beta (R_\beta^2 - \Pi^2)}{2v}}{p + \dfrac{\alpha_s q K_\beta}{3v^2}(R_\beta^3 - \Pi^3) - \dfrac{\alpha_s q K_\beta \Pi}{2v^2}(R_\beta^2 - \Pi^2) - \dfrac{\alpha_s q R_s}{v}} - \frac{K_\beta \gamma}{2v} \,. \tag{34}$$

6.5 *Effect of Assault Speed and Percentage Force Split*[1]

After considerable rearrangement of (33),

$$d_o = pM - N + \frac{T_1}{v} + \frac{T_2}{v^2} + \frac{T_3}{v^3} \,, \tag{35}$$

where

$$T_1 = \frac{pK_\beta N\gamma}{2} - \alpha_s q R_s M + \frac{(R_\beta^2 - \Pi^2)K_\beta N}{2}$$

$$T_2 = \frac{\alpha_s q K_\beta (R_\beta^3 - \Pi^3)M}{3} - \frac{\alpha_s q K_\beta \Pi M (R_\beta^2 - \Pi^2)}{2} - \frac{\alpha_s q R_s K_\beta N\gamma}{2}$$

---

[1] Some recent results indicate that "bang-bang" controls should be applied to v and p so as to maximize d_o. These results will be described in a later report.

$$T_3 = \frac{\alpha_s q K_\beta (R_\beta^3 - \Pi^3) M}{3} - \frac{\alpha_s q K_\beta \Pi K_\beta N \gamma (R_\beta^2 - \Pi^2)}{4}$$

$$= \frac{\alpha_s q K^2 N \gamma \psi}{2} ,$$

where

$$\psi = R_s^2 \left(\frac{R_\beta}{2} - \frac{R_s}{6} \right) .$$

Taking the partial with respect to the assault speed

$$\frac{\partial d_o}{\partial v} = - \frac{T_1}{v} - \frac{2T_2}{v^3} - \frac{3T_3}{v^4} . \tag{36}$$

To evaluate the behavior of (36), one needs to know the algebraic signs of T_1, T_2, and T_3. It is easily shown that T_3 is always negative. Hence, $- \frac{3T_3}{v^4}$ is always positive, since by the coordinate system assumes that v will be a negative quantity.

It is beneficial to know the conditions under which d_o can be increased with increased velocity. By inspection, if $T_2 > 0$ and $T_1 < 0$, $\frac{\partial d_o}{\partial v}$ will be positive. $T_2 > 0$ implies that $\alpha_s q K_\beta M \psi > \frac{\alpha_s R_s K_\beta N \gamma q}{2}$ or, after some rearrangement, that

$$\frac{M}{N} > \frac{\gamma}{R_s \left(R_\beta - \frac{R_s}{3} \right)} . \tag{37}$$

$T_1 < 0$ implies that $\frac{K_\beta N}{2}[p\gamma + (R_\beta^2 - \Pi^2)] < \alpha_s q R_s M$, or, after some rearrangement, that

$$\frac{\cdot}{N} < \frac{R_s[p\gamma + (R_s^2 - \Pi^2)]}{2\alpha_s q R_s} \quad . \tag{38}$$

[illegible] (37) and (38) are satisfied, $\frac{\partial d_o}{\partial v}$ is positive and [illegible] increased by increasing the velocity of the attack. [illegible] under which the reverse is true are more difficult [illegible].

The conditions under which an assault speed exists, which [illegible] or maximizes d_o, are given by[1]

$$\frac{\partial^2 d_o}{\partial v^2} = \frac{2T_1}{v^3} + \frac{6T_2}{v^4} + \frac{12T_3}{v^5} \tag{39}$$

$$T_3 < 0 \Longrightarrow \frac{12T_3}{v^5} > 0 \tag{40}$$

$$T_2 > 0 \Longrightarrow \frac{6T_2}{v^4} > 0 \tag{41}$$

$$T_1 < 0 \Longrightarrow \frac{2T_1}{v^3} > 0 \quad . \tag{42}$$

[illegible] (40), (41), and (42) suggest that the second [illegible] positive if (37) and (38) hold. Hence, the speed [illegible] (..) equal to zero and solving would be the one [illegible] $\frac{v}{\cdot}$ in the specified interval.

Consider next the influence of the force split p. After some manipulation, (33) may be put into the form

$$d_o = Ap + AB(1 - p) + C,$$

where

$$A = \frac{M}{N} + \frac{K_\beta \gamma}{2v} = \frac{M_s}{N}$$

$$B = \frac{\alpha_s K_\beta}{3v^2}(R_\beta^3 - \Pi^3) - \frac{\alpha_s K_\beta \Pi}{2v^2}(R_\beta^2 - \Pi^2) - \frac{\alpha_s R_s}{v}$$

$$C = \frac{K_\beta (R_\beta^2 - \Pi^2)}{2v} - 1 .$$

Hence, $\frac{\partial d_o}{\partial p} = A - AB =$ constant. To check the extreme conditions, one can see that d_o is at a maximum when $p = 1$ if

$$C + A > AB + C \quad \text{or} \quad B < 1 . \tag{43}$$

For the reverse of (43), d_o will be at a maximum when $p = 0$.

PART D

HETEROGENEOUS-FORCE DIFFERENTIAL MODELS

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The preceding parts of the report described efforts to obtain solutions for the differential equation description of homogeneous-force battles. These descriptions were simplifications of the general variable coefficient differential equation model of heterogeneous-force battles.[1] In this part of the report we present solutions and solution procedures for simplified forms of the differential equation description of heterogeneous-force battles. Chapter 1 contains the development of solutions for the heterogeneous-force battle model when the attrition rates are constant and a "zero-one" allocation policy is employed. Chapter 2 contains a description of our efforts to develop optimal allocation strategies in context of the heterogeneous-force model. Chapter 3 describes a simplified numerical solution procedure for the general heterogeneous-force model and a computer program for performing the computations.

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[1] See equations 1 and 2 in Chapter 2, Part A.

Chapter 1

CONSTANT ATTRITION-COEFFICIENT MODEL

Stanley Sternberg

1.1 Introduction and Notation

In this chapter we shall discuss the solution of the following differential equations representing a heterogeneous-force battle:

$$\frac{dm_i}{dt} = - \sum_{j=1}^{J} \beta_{ji} h_{ji} n_j \qquad [1]$$

$$i = 1,2,\ldots,I$$

$$m_i(t=0) = M_i$$

$$\frac{dn_j}{dt} = - \sum_{i=1}^{I} \alpha_{ij} e_{ij} m_i \qquad [2]$$

$$j = 1,2,\ldots,J$$

$$n_j(t=0) = N_j \ ,$$

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where

α_{ij} = *the attrition rate*--the rate at which an individual system in the i^{th} Blue group attrits live j^{th} group Red targets when it is firing at them.

e_{ij} = *the allocation factor*--the proportion of i^{th} Blue group systems assigned to fire on j^{th} Red group targets. These are assumed to be either zero or one for any i,j pair.[1]

Similar definitions apply to β_{ji} and h_{ji}. Equations 1 and 2 are similar to those presented in Part A of the report except (a) perfect intelligence is assumed for both sides and (b) the attrition rates and allocation factors are not range dependent and are treated as constant.

To facilitate the study of [1] and [2] we introduce the row vectors $\bar{m}$, $\bar{n}$, $\bar{M}$, and $\bar{N}$, whose elements are the m_i, n_j, M_i, and N_j, respectively. The derivatives of $\bar{m}$ and $\bar{n}$ are appropriately defined as the row vectors

$$\frac{d\bar{m}}{dt} = \left(\frac{dm_1}{dt}, \ldots, \frac{dm_I}{dt}\right)$$

and

$$\frac{d\bar{n}}{dt} = \left(\frac{dn_1}{dt}, \ldots, \frac{dn_J}{dt}\right) .$$

[1]The value of this zero-one allocation policy is discussed in Chapter 2 of this part.

The matrices A and B are defined as

$$A = (a_{ij}) = (e_{ij}\alpha_{ij})$$

$$B = (b_{ji}) = (h_{ji}\beta_{ji}) .$$

It follows that equations 1 and 2 can be rewritten

$$\frac{d\bar{m}}{dt} = -\bar{n}B, \quad \bar{m}(t=0) = \bar{M} \tag{3}$$

$$\frac{d\bar{n}}{dt} = -\bar{m}A, \quad \bar{m}(t=0) = \bar{N} . \tag{4}$$

An alternate form to (3) and (4) that will be very useful is defined in terms of the row vectors

$$\bar{z} = (\bar{m}, \bar{n}), \quad \bar{q} = (\bar{M}, \bar{N}) ,$$

$$\frac{d\bar{z}}{dt} = \left(\frac{d\bar{m}}{dt}, \frac{d\bar{n}}{dt}\right)$$

and the matrix

$$C = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} .$$

The constant-coefficient, heterogeneous-force model of the combat process may then be represented by the single matrix equation

$$\frac{d\bar{z}}{dt} = -\bar{z}C, \quad \bar{z}(0) = \bar{q} . \tag{5}$$

The solution of equation 5 is a vector whose elements are functions of t. It will be called continuous if its elements are continous functions of t in the interval of interest. Similar definitions apply to matrix functions.

1.2 Existence and Uniqueness of Solutions of Linear Systems

A unique solution exists to equation 5, as demonstrated by the following basic theorem:

Theorem 1

If $A(t)$ is continuous for $t \geq 0$, there is a unique solution to the vector differential equation

$$\frac{d\bar{x}}{dt} = \bar{x}A(t), \quad \bar{x}(0) = \bar{q} . \tag{6}$$

This solution exists for $t \geq 0$, and may be written in the form

$$\bar{x} = \bar{q}X(t) , \tag{7}$$

where $X(t)$ is the unique matrix satisfying the matrix differential equation

$$\frac{dX}{dt} = XA(t), \quad X(0) = I , \tag{8}$$

where I is the identity matrix.

The proof of theorem 1, as presented by Bellman, is given in Appendix D, 1, 1. Our particular problem is concerned with the case in which $A(t)$ is a constant matrix.

1.3 *The Matrix Exponential*

In the scalar case, the equation

$$\frac{dx}{dt} = ax, \quad x(0) = q \tag{9}$$

has a solution

$$x = qe^{at} . \tag{10}$$

The analogous solution of the matrix equation

$$\frac{d\bar{x}}{dt} = \bar{x}A, \quad \bar{x}(0) = \bar{q} \tag{11}$$

has the form

$$\bar{x} = \bar{q}e^{At} . \tag{12}$$

By analogy with the scalar case, we define the matrix exponential by the infinite series

$$e^{At} = I + At + \ldots + \frac{A^n t^n}{n!} + \ldots . \tag{13}$$

This matrix series exists for all A for any fixed value of t, and for all t for any fixed A. It converges uniformly for finite t. A proof of convergence is given in Appendix D, 1, 2.

To show that equation 12 is the unique solution to matrix differential equation 11 requires that e^{At}, as defined by (13), satisfies

$$\frac{de^{At}}{dt} = e^{At}A , \tag{14}$$

$$e^{At} = I \text{ for } t = 0 ,$$

as required by theorem 1. The validity of equation 14 is obvious.

1.4 *Similarity, Diagonalizability, and Jordan Normal Form*

Since the solution of the differential equation 5 can be written immediately as

$$\bar{z} = \bar{q}e^{-tC} , \tag{15}$$

our problem reduces to evaluating the matrix exponential e^{-tC}. The infinite series given by (13), of course, is always available, but not very attractive. Our object is to write equation 15 in a closed form which will lend itself to rapid computation.

The solution is facilitated by the fact that the attrition matrix C has a very special form. Recall that

$$C = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} \tag{16}$$

and that

$$(a_{ij}) = (e_{ij}\alpha_{ij}) \tag{17}$$

$$(b_{ji}) = (h_{ji}\beta_{ji}) \ . \tag{18}$$

When the fraction of type-i components assigned to opposing type-j components is either 0 or 1, or vice versa, the matrix C is said to be "row elemental."

Definition 1

A real matrix A is "row elemental" if each of its rows contains exactly one nonzero element. Similarly, A is "column elemental" if each of its columns contains exactly one nonzero element.

The concept of the similarity of matrices is used in the development of our solution procedure. A square matrix A is said to be similar to a square matrix B if there exists a nonsingular matrix R such that

$$A = R^{-1}BR \ . \tag{19}$$

Of particular concern is the situation where A is similar to a diagonal matrix D, i.e.,

$$A = R^{-1}DR \ , \tag{20}$$

and we say that the matrix A is diagonalizable.

The reason for this particular interest becomes apparent when we note that

$$A^n = (R^{-1}DR)(R^{-1}DR) \ \ldots \ (R^{-1}DR) \tag{21}$$

or

$$A^n = R^{-1}D^nR \ . \tag{22}$$

Now the matrix exponential e^{-tA} can be written in terms of the powers of D as

$$e^{-tA} = I - tR^{-1}DR + \frac{t^2R^{-1}D^2R}{2!} - \frac{t^3R^{-1}D^3R}{3!} + \ldots \qquad (23)$$

or

$$e^{-tA} = R^{-1}\left\{I - tD + \frac{t^2D^2}{2!} - \frac{t^3D^3}{3!} + \ldots\right\} R \,. \qquad (24)$$

But the diagonal matrix D raised to the n^{th} power is simply

$$D^n = \begin{pmatrix} d_{11}^n & & & 0 \\ & d_{22}^n & & \\ & & \ddots & \\ 0 & & & d_{nn}^n \end{pmatrix} . \qquad (25)$$

Thus, the bracketed expression of equation 24 is in actuality of the form

$$I - tD + \frac{t^2D^2}{2!} - \frac{t^3D^3}{3!} + \ldots = \begin{pmatrix} e^{-td_{11}} & & & 0 \\ & e^{-td_{22}} & & \\ & & \ddots & \\ 0 & & & e^{-td_{nn}} \end{pmatrix} . \qquad (26)$$

Therefore, assuming that A is similar to a diagonal matrix D, the matrix exponential may be evaluated from the expression

$$e^{-tA} = R^{-1}\begin{pmatrix} e^{-td_{11}} & & & 0 \\ & e^{-td_{22}} & & \\ & & \ddots & \\ 0 & & & e^{-td_{nn}} \end{pmatrix} R \,. \tag{27}$$

In the case where the attrition matrix is similar to a diagonal matrix, the analysis is now quite clear even though the actual determination of R and D has not as yet been discussed. Unfortunately, however, the attrition matrix C is not generally diagonalizable.

The situation is remedied somewhat if we relax our assumption that C is similar to a diagonal matrix to the condition that C be similar to a matrix in Jordan normal form.

A matrix J is said to be in Jordan normal form if it is zero everywhere except for submatrices along its diagonal, all of which are Jordan blocks. If $\boxed{J_1}$, $\boxed{J_2}$, ..., $\boxed{J_m}$ are Jordan blocks, then the matrix

$$J = \begin{pmatrix} \boxed{J_1} & 0 & \cdots & 0 \\ 0 & \boxed{J_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \boxed{J_m} \end{pmatrix} \tag{28}$$

is in Jordan normal form. A Jordan block is a square matrix of the form

$$\boxed{J_k} = \begin{pmatrix} \lambda_k & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_k \end{pmatrix} . \tag{29}$$

That is, it contains a sequence of 1's along its "superdiagonal," while everywhere else it is zero, except possibly along its diagonal, which contains a sequence of identical, not necessarily real, numbers, λ_k. Thus, the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

is a Jordan block, so is the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and the matrix of a single element is also a Jordan block.

Our interest in Jordan normal matrices will be restricted to those having Jordan blocks with zero elements along the diagonal.

Definition 2

A "zero Jordan block," denoted $\boxed{J_k^0}$, is a Jordan block with λ_k equal to zero.

A Jordan normal matrix whose diagonal consists entirely of zero Jordan blocks and/or diagonal matrices will receive particular attention.

Definition 3

A "zero Jordan normal matrix," denoted J^0, is a matrix of the form

$$\begin{pmatrix} D_1 & & & & & 0 \\ & \ddots & & & & \\ & & D_L & & & \\ & & & \boxed{J_1^0} & & \\ & & & & \ddots & \\ 0 & & & & & \boxed{J_M^0} \end{pmatrix},$$

where the D_k are diagonal matrices and the $\boxed{J_k^0}$ are zero Jordan blocks.

We now state our main result in the form of a theorem and demonstrate its application to the solution of the heterogeneous-force equations in the next section.

Theorem 2

If A is a square, row-elemental matrix, then A is similar to a zero Jordan normal matrix.

The proof of theorem 2 is given in Sections 6 through 10.

1.5 Solution of the Heterogeneous-Force Differential Equations

In this section we assume that theorem 2 is true and demonstrate its consequences. We have given that

$$C = R^{-1}J^0R \; . \tag{30}$$

where C is a row-elemental attrition matrix. Then,

$$e^{-tC} = R^{-1}\left(I - tJ^0 + \frac{t^2J^{0^2}}{2!} - \ldots\right)R \; . \tag{31}$$

The powers of J^0 are

$$J^{0^n} = \begin{pmatrix} D_1^n & & & & & & 0 \\ & \ddots & & & & & \\ & & D_L^n & & & & \\ & & & \boxed{J_1^0}^n & & & \\ & & & & \ddots & & \\ 0 & & & & & & \boxed{J_M^0}^n \end{pmatrix}, \tag{32}$$

where the D_k^n are

$$D_k^n = \begin{pmatrix} {d_{11}}_k^n & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & {d_{mm}}_k^n \end{pmatrix} . \qquad (33)$$

The powers of a zero Jordan block are quite easy to compute as illustrated in the following example:

$$\boxed{J_k^0} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \boxed{J_k^0}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \boxed{J_k^0}^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\boxed{J_k^0}^n$ is the zero matrix for $n > 3$. In other words, if $\boxed{J_k^0}$ is a zero Jordan block of order N, then $\boxed{J_k^0}^n$ is nonzero only for $n < N$, and the n^{th} power of $\boxed{J_k^0}$ is zero everywhere except for a superdiagonal of 1's displaced n times from the main diagonal.

The bracketed term of equation 31 is therefore the matrix function

$$F(-t) = \begin{pmatrix} E_1(-t) & & & & & & 0 \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & E_L(-t) & & & \\ & & & & T_1(-t) & & \\ & & & & & \cdot & \\ 0 & & & & & & T_M(-t) \end{pmatrix} , \qquad (34)$$

where

$$E_k(-t) = \begin{pmatrix} e^{-td_{11_k}} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & \cdot & \\ 0 & & & e^{-td_{mm_k}} \end{pmatrix} \qquad (35)$$

and

$$T_k(-t) = \begin{pmatrix} 1 & -t & t^2/2! & -t^3/3! & \cdots & (-1)^{m+1}t^m/m! \\ 0 & 1 & -t & t^2/2! & \cdots & (-1)^m t^{m-1}/(m-1)! \\ 0 & 0 & 1 & -t & \cdots & \\ 0 & 0 & 0 & 1 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix} \qquad (36)\ .$$

The solution of the heterogeneous-force differential equations is therefore

$$(\bar{m}\ \bar{n}) = (\bar{M}\ \bar{N})R^{-1}F(-t)R \ . \qquad (37)$$

1.6 Assignment Chains and Cycles

When we examine the actual assignments which could arise during a heterogeneous-force battle process, we recognize two distinctly different situations which lead to two different kinds of time solutions to the model differential equations. In the first situation we have "cyclic assignments." For example, m_1 is assigned to n_3, who is assigned to m_4, who is

assigned to n_2, who completes the cycle by being assigned to m_1. Of course, here we are speaking about 0,1 assignments where each component group is assigned to only a single opposing component group. In the second situation m_3 might be assigned to n_1, who is assigned to m_2, who in turn is assigned to one of the components in the preceding cycle. Thus m_3 is an unassigned component and suffers no attrition, while n_1 and m_2 form part of the "chain" headed by m_3.

It should come as no surprise that the cyclic assignments are directly related to the exponential terms of the time solution shown in equation 37, while each assignment chain gives rise to a submatrix T_k. The complicated interrelations between the many possible assignment cycles and assignment chains are manifested within the similarity transform matrix, R .

We now define the above concepts in a more formal manner with respect to the attrition matrix, C. Let C be a row elemental matrix of order N and let W be the set of N subscript pairs of the nonzero elements of C,

$$W = \{(1,t_1), (2,t_2), \ldots, (N,t_N)\} , \qquad (38)$$

where the nonzero element on the i^{th} row of C occurs on the t_i'th column of C. Suppose that n-ordered sequence S_m of length m can be formed from a subset of W

$$S_m = (i_1,t_{i_1}), (i_2,t_{i_2}), \ldots, (i_m,t_{i_m}) \qquad (39)$$

such that

$$t_{i_k} = i_{k+1} \tag{40}$$

for $k = 1,\ldots,m-1$. Such a sequence is said to form a "subscript chain." For example,

$$S_4 = (4,3)\ (3,2),\ (2,5)\ (5,1)$$

is a subscript chain of length 4. The elements of C whose subscripts form the subscript chain are said to form an "element chain," or simply a "chain" of length m.

If S_m is a subscript chain and if

$$t_{i_m} = i_1 \quad , \tag{41}$$

then the subscript chain is said to form a "subscript cycle." The elements of C whose subscripts form the subscript cycle are said to form an "element cycle," or simply a "cycle" of length n, denoted C_n. An example of a cycle of length three is

$$C_3 = (c_{2,5}),\ (c_{5,3})\ (c_{3,2}) \ .$$

A nonzero diagonal element of C forms a cycle of length one.

The following properties concerning the row-elemental matrix C are sufficiently obvious as to be stated largely without proof:

Property 1 All nonzero elements of C belong to either cycles or chains, or both. To avoid the ambiguity of the latter case, we will say that an element

belongs to a chain if and only if it does not belong to a cycle.

Property 2 An element can belong to at most one cycle. For if two different cycles share common elements, then there is one common element, say $c_{j,k}$, which is followed by nonzero elements c_{k,t_k} and $c_{k,t_{k'}}$, $k \neq k'$, of different cycles. But this contradicts our premise that C is row elemental.

Property 3 There are no cycles of length one in C. Similarly, there are no chains of length one. (C has a zero diagonal.)

Property 4 All cycles of C are of even length.

Property 5 Let A be a row-elemental matrix, all of whose nonzero elements form a single cycle. Then A is also column elemental.

Definition 4

A "cyclic matrix" is a row-elemental matrix whose nonzero elements form a single cycle.

In proving that a row-elemental attrition matrix C is similar to a zero Jordan normal matrix (theorem 2), we will first show that C is similar to a "cyclic normal matrix."

Definition 5 A row-elemental matrix of the form

$$Q = \begin{array}{c} \begin{pmatrix} C_1 & 0 \dots & 0 & 0 \\ 0 & C_2 \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 \dots & C_m & 0 \\ D_1 & D_2 \dots & D_m & G \end{pmatrix} \end{array} \begin{array}{l} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} t \text{ rows} \\ \left.\vphantom{a}\right\} n-t \text{ rows} \end{array} \qquad (42)$$

is said to be "cylic normal" if $C_1, C_2, \ldots, C_m$ are cyclic submatrices and the nonzero elements of submatrix G do not form any cycles.

Theorem 3

Let A be a square, row-elemental matrix. Then A is similar to a cyclic normal matrix Q, i.e.,

$$A = P^{-1}QP, \tag{43}$$

where the similarity transform P is a permutation matrix.

Before proceeding with the proof of theorem 3, we introduce a few definitions concerning permutations.

Definition 6

A "permutation of degree n" is the operation of changing the order of n given distinct objects. If the n distinct objects are the numbers $1,\ldots,n$, a permutation is the replacement of one arrangement $(\lambda_1,\ldots,\lambda_n)$ of $(1,\ldots,n)$ by a second arrangement $(\mu_1,\ldots,\mu_n)$. We represent this permutation by

$$\pi = \begin{pmatrix} \lambda_1,\ldots,\lambda_n \\ \mu_1,\ldots,\mu_n \end{pmatrix} .$$

We frequently say that the permutation π transforms λ_i into μ_i or that μ_i is the image of λ_i under π, i.e., $\pi(\lambda_i) = \mu_i$.

Definition 7

The "product" $\sigma\pi$ of two permutations π and σ is the permutation resulting from first carrying out π and then σ. Thus if

$$\pi = \begin{pmatrix} 1,\ldots,n \\ \lambda_1,\ldots,\lambda_n \end{pmatrix} \qquad \sigma = \begin{pmatrix} \lambda_1,\ldots,\lambda_n \\ \mu_1,\ldots,\mu_n \end{pmatrix} ,$$

then

$$\sigma\pi = \begin{pmatrix} 1,\ldots,n \\ \mu_1,\ldots,\mu_n \end{pmatrix} .$$

It follows that the inverse of

$$\pi = \begin{pmatrix} \lambda_1,\ldots,\lambda_n \\ \mu_1,\ldots,\mu_n \end{pmatrix}$$

is the permutation

$$\pi^{-1} = \begin{pmatrix} \mu_1,\ldots,\mu_n \\ \lambda_1,\ldots,\lambda_n \end{pmatrix} .$$

Definition 8

With each permutation π of degree n is associated the n-by-n matrix P defined by the equation

$$(P_{ij}) = \begin{cases} 1 & \text{whenever } i = \pi(j) \\ 0 & \text{otherwise} \end{cases}$$

for $i, j = 1,\ldots,n$. Thus, if π is the permutation

$$\pi = \begin{pmatrix} 1,\ldots,n \\ \lambda_1,\ldots,\lambda_n \end{pmatrix},$$

the first column of P contains a 1 in the $\lambda_1{}^{th}$ row, the second column of P contains a 1 in the $\lambda_2{}^{th}$ row, and so forth, while all the remaining elements are equal to zero. A matrix of this type is called a "permutation matrix."

It follows from definitions 7 and 8 that, if P is a permutation matrix associated with the permutation π, then P^{-1} is the permutation matrix associated with π^{-1}.

A permutation π may be performed on the *rows* of a square matrix A by *premultiplying* A by the permutation matrix P associated with π. Let pa_{ij} denote the elements of the matrix product PA, then

$$pa_{ij} = \sum_k p_{ik}\, a_{kj}\,. \tag{44}$$

There is only one nonzero element, p_{ik*}, in the i^{th} row of P. In particular, $p_{ik*} = 1$ where $k* = \pi^{-1}(i)$. Hence, the nonzero product in the summation over k is pa_{ij} if and only if $i = \pi(k)$.

A permutation π may be performed on the *columns* of a square matrix A by *postmultiplying* A by the inverse of the permutation matrix P associated with π. Transposing A replaces the rows of A by the columns of A. Premultiplying A^T by P permutes the rows of A^T. Transposing the matrix product

PA^T yields AP^T, which replaces the columns of A^T by the permuted rows of A^T; hence, A has been permuted columnwise according to π. Since the columns of P are mutually orthogonal, normal vectors, $P^T = P^{-1}$. The operation of simultaneously interchanging (permuting) the rows and columns of a square matrix A according to a permutation π is therefore accomplished by the matrix operation PAP^{-1}.

Simultaneous row and column interchanges are all that are required to put a square, row-elemental matrix into cyclic normal form. Let A be square and row-elemental and let W be the set of subscript pairs of the nonzero elements of A

$$W = \{(1,t_1),(2,t_2),\ldots,(n,t_n)\} \quad .$$

Let $C_1,\ldots,C_m$ be subsets of W of subscript pairs forming cycles in A. In particular, cycle k consists of the ℓ subscript pairs

$$C_k = \{(k_1,t_{k_1}),\ldots,(k_\ell,t_{k_\ell})\} \quad .$$

Let π_k be the permutation

$$\pi_k = \begin{pmatrix} k_1 & k_2 \ldots k_\ell \\ r & r+1 \ldots r+\ell \end{pmatrix}$$

and P_k its associated permutation matrix. Then the operation P_kA interchanges the rows $k_1,\ldots,k_\ell$ of A with rows $r,\ldots,r+\ell$ of A. The operation AP_k^{-1} interchanges columns $k_1,\ldots,k_\ell$ with columns $r,\ldots,r+\ell$ of A. But columns $k_1,\ldots,k_\ell$ precisely

contain the elements of cycle C_k because $\{t_{k_1},\ldots,t_{k_\ell}\}$ is identical to $\{k_1,\ldots,k_\ell\}$. Therefore, $P_kAP_k^{-1}$ moves the elements of cycle C_k into the square submatrix C_k on the diagonal of Q. The fact that A is row elemental insures that C_k will contain only the nonzero elements of cycle C_k.

Simultaneous row and column interchanges of the type just described on all cycles of A are accomplished by the transformation

$$P_mP_{m-1}\cdots P_1A\,P_1^{-1}\cdots P_{m-1}^{-1}\,P_m^{-1}\quad, \tag{45}$$

or simply

$$PAP^{-1}\ , \tag{46}$$

where $P = P_mP_{m-1}\cdots P_1$. If j* is the column subscript of a nonzero element a* of A not belonging to a cycle of A, but with j* contained in the set of column indices of cycles of A, then i*, the row subscript of a*, cannot belong to any set of row indices of cycles, or a* would itself belong to a cycle. The similarity transformation PAP^{-1} therefore carries column j* into the first t columns of Q, but carries row i* into the last n-t rows of Q. The submatrices D_k, therefore, are made up of nonzero elements sharing rows with the elements of C_k. All submatrices to the left and right of C_k must be zero. Finally, the matrix G is composed of interchanged nonzero elements of A not previously sharing columns with elements belonging to cycles.

The cyclic normal matrix Q is therefore obtained from A as

$$Q = PAP^{-1} \tag{47}$$

or

$$A = P^{-1}QP \quad . \tag{48}$$

1.7 Jordan Normal Matrices

The Jordan normal matrix was previously discussed in Section 1.4. Theorem 2 stated without proof that, if A is a square, row-elemental matrix, then A is similar to a particular Jordan normal form, called a zero Jordan normal matrix. The zero Jordan normal form was defined in terms of submatrices along its diagonal, which were stated to be either diagonal submatrices or zero Jordan blocks.

In theorem 3 it was shown that a square, row-elemental matrix A is similar to a cyclic normal matrix Q. We begin at this point to demonstrate that Q is similar to a zero Jordan normal matrix J^o. Since

$$A = P^{-1}QP \tag{49}$$

by theorem 3, and as we will demonstrate,

$$Q = S^{-1}J^oS \quad , \tag{50}$$

the proof of theorem 2 immediately follows as

$$A = P^{-1}S^{-1}J^cSP \tag{51}$$

or

$$A = R^{-1}J^{o}R \ , \tag{52}$$

where R = SP .

Our particular result follows as a consequence of the following well-known theorem of linear algebra:

Theorem 4

Let A be an arbitrary square matrix. Then there exists a nonsingular matrix T such that

$$A = T^{-1}JT \ , \tag{53}$$

where J is a Jordan normal matrix.

Since the proof of this theorem is quite detailed and is given by Franklin (1968), we shall avoid the proof but we will discuss its consequences.

We write the N×N Jordan normal matrix J as

$$J = \begin{pmatrix} \boxed{J_1} & & 0 \\ & \ddots & \\ 0 & & \boxed{J_M} \end{pmatrix} ,$$

where the k^{th} Jordan block

$$\boxed{J_k} = \begin{pmatrix} \lambda_k & 1 & \ldots & 0 & 0 \\ 0 & \lambda_k & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & \lambda_k & 1 \\ 0 & 0 & \ldots & 0 & \lambda_k \end{pmatrix}$$

is of order n_k; hence, $n_1 + \cdots + n_M = N$. Let the rows of T be denoted by the row vectors $\overline{T}_1, \ldots, \overline{T}_N$ and let the submatrix T_k of T consist of the M_k rows $\overline{T}_{s_k+1}, \ldots, \overline{T}_{s_k+n_k}$, where $s_k = n_1 + \cdots + n_{k-1}$. Premultiplying both sides of equation 53 by T yields

$$TA = JT \,, \tag{54}$$

which is equivalent to the system

$$T_k A = \boxed{J_k}\, T_k \,; \quad k = 1, \ldots, M. \tag{55}$$

Expanding (55) shows that

$$\begin{pmatrix} \overline{T}_{s_k+1} \\ \vdots \\ \overline{T}_{s_k+n_k} \end{pmatrix} A = \begin{pmatrix} \lambda_k & 1 & \ldots & 0 & 0 \\ 0 & \lambda_k & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_k & 1 \\ 0 & 0 & \ldots & 0 & \lambda_k \end{pmatrix} \begin{pmatrix} \overline{T}_{s_k+1} \\ \vdots \\ \overline{T}_{s_k+n_k} \end{pmatrix} \tag{56}$$

or

$$\begin{aligned} \overline{T}_{s_k+1} A &= \lambda_k \overline{T}_{s_k+1} + \overline{T}_{s_k+2} \\ &\vdots \\ \overline{T}_{s_k+n_k-1} A &= \lambda_k \overline{T}_{s_k+n_k-1} + \overline{T}_{s_k+n_k} \\ \overline{T}_{s_k+n_k} A &= \lambda_k \overline{T}_{s_k+n_k} \,, \end{aligned} \tag{57}$$

which, on collecting common terms, is

$$\overline{T}_{s_k+1}(A - \lambda_k I) = \overline{T}_{s_k+2}$$
$$\vdots$$
$$\overline{T}_{s_k+n_k-1}(A - \lambda_k I) = \overline{T}_{s_k+n_k} \tag{58}$$
$$\overline{T}_{s_k+n_k}(A - \lambda_k I) = 0 \quad .$$

Now, substituting the last equation into the next to last and so forth yields

$$\overline{T}_{s_k+1}(A - \lambda_k I)^{n_k} = 0$$
$$\vdots$$
$$\overline{T}_{s_k+n_k-1}(A - \lambda_k I)^2 = 0 \tag{59}$$
$$\overline{T}_{s_k+n_k}(A - \lambda_k I) = 0 \quad .$$

The vector $\overline{T}_{s_k+n_k}$ is by definition an eigenvector of A corresponding to the eigenvalue λ_k. The eigenvalue λ_k has multiplicity n_k in A and the remaining n_k-1 vectors $\overline{T}_{s_k+1},\ldots,$ $\overline{T}_{s_k+n_k=1}$ are called "latent eigenvectors" of A corresponding to eigenvalue λ_k. The $\overline{T}_i$ are necessarily linearly independent as the matrix T is nonsingular.

Both eigenvectors and latent eigenvectors are examples of what Franklin calls "principal vectors." He states that a zero or nonzero vector $\overline{p}$ is a principal vector of grade $g \geq 0$ belonging to the eigenvalue λ_i if

$$(\lambda_i I - A)^g \overline{p} = 0$$

and if there is no smaller non-negative integer $\gamma < g$ for which $(\lambda_i I - A)^\gamma \overline{p} = 0$.

The vector $\bar{p} = 0$ is the principal vector of grade 0. The eigenvectors are the principal vectors of grade 1. Herein we shall speak in terms of principal vectors of grade g rather than this ambiguous latent eigenvectors and restrict the term eigenvector for a principal vector of grade 1.

In the next section we discuss the eigenvalues and eigenvectors of the cyclic submatrices on the diagonal of the cyclic normal matrix Q.

1.8 *Eigenvalues of Cyclic Matrices*

We begin by stating our main result.

Theorem 5

Let C be an N-by-N cyclic matrix: (i) Then C is similar to a diagonal matrix; (ii) Let p be the product of the N nonzero elements of C, then C has N distinct eigenvalues equal to the N^{th} roots of $(-1)^N p$.

We need only prove part (ii), for part (i) follows immediately from the following theorem.

Theorem 6

[Mirsky, (1961), p. 296] If the N eigenvalues of the N-by-N matrix A are distinct, then A is similar to a diagonal matrix.

We will need the following definitions in proving part (ii) of theorem 5.

Definition 9

(i) Let A be an M-by-N matrix. If $K \leq M$ and $L \leq N$, then any K rows and L columns of A determine a K-by-L "submatrix" of A. (ii) The determinant of a K-by-K submatrix of A is called a "K-rowed minor" of A.

When A is square, the following definition is relevant.

Definition 10

A "principal submatrix" of A is a submatrix whose diagonal is part of the diagonal of A. The determinant of K-by-K principal submatrix of A is called a "K-rowed principal minor" of A.

A K-by-K principal submatrix is obtained from the N-by-N matrix A by deleting N-K rows of A and the corresponding columns, i.e., rows and columns having like indices.

Finally, so there is no misunderstanding:

Definition 11

Let $A = (a_{ij})$ be an N-by-N matrix and λ a scalar variable. The "characteristic polynomial" of A is the polynomial $g(\lambda)$ given by

$$g(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1N} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2N} \\ -a_{N1} & -a_{N2} & \cdots & \lambda - a_{NN} \end{vmatrix} . \tag{60}$$

The characteristic equation of A is the equation $g(\lambda) = 0$. Its roots are the eigenvalues (or characteristic roots) of A.

The polynomial $g(\lambda)$ is of degree N. Its leading term is λ^N. The remaining coefficients can be determined by the following theorem:

Theorem 7

[Mirsky, p. 197] For $0 \leq r < N$, the coefficients of λ^r in the characteristic polynomial $g(\lambda)$ of A is equal to $(-1)^{N-r}$ times the sum of all (N-r)-rowed principal minors of A.

The principal minors of a cyclic matrix may be evaluated by the following theorem:

Theorem 8

Let C be an N-by-N cyclic matrix. Then for $0 < r < N$, all (N-r)-rowed principal minors of C are zero.

The proof of theorem 8 follows. Let C' denote an (N-r)-rowed principal submatrix of C where $0 < r < N$. Since C is both w- and column-elemental, its principal submatrices, formed by deleting corresponding rows and columns of C, must necessarily contain fewer nonzero elements than C. As the nonzero elements of C form a single cycle, the nonzero elements of C' cannot therefore form any cycles.

If we denote the n^{th} power of C' as $C'^n = (c'^n_{ij})$, then we can write

$$c'^n_{ij} = \sum_{k_1=1}^{N-r} \sum_{k_2=1}^{N-r} \cdots \sum_{k_{n-2}=1}^{N-r} \sum_{k_{n-1}=1}^{N-r} \Big[c'_{ik_1} c'_{k_1k_2} \cdots c'_{k_{n-2}k_{n-1}} c'_{k_{n-1}j} \Big] \quad . \tag{61}$$

Note that the elements in the summand above form an n-element chain. Now suppose $n = N$. The elements in the summand chain cannot all be nonzero, for if they were, they would necessarily have to be distinct; otherwise, the N-element chain would contain a cycle. But C' cannot contain N distinct nonzero elements simply because it has fewer than N nonzero elements. Therefore, c'^{N}_{kj} is zero and C'^{N} is the zero matrix. The proof of the theorem follows from the fact that

$$|C'|^n = |C'^n| \ .$$

For $r = 0$, the single (N-r)-rowed principal minor of C is simply the determinant of C, which is given by $(-1)^{N+1}$ times the product of the nonzero elements of C. Letting p denote this product, the characteristic equation of C is then, by theorems 7 and 8,

$$\lambda^N + (-1)^{N+1} p = 0 \quad . \tag{62}$$

This result proves part (ii) of theorem 5.

1.9 Eigenvectors of Cyclic Matrices

Before we can transform a cyclic matrix to its diagonal form, we must compute the necessary transformation matrix. This is accomplished by computing its eigenvectors.

Theorem 9

(Mirsky, 1961, p. 293). If $X_1, \ldots, X_n$ are linearly independent eigenvectors of an n-by-n matrix A, and S

is the (nonsingular) matrix having $X_1, \ldots, X_n$ as its columns, then $S^{-1}AS$ is a diagonal matrix.

Eigenvectors corresponding to distinct eigenvalues of A are linearly independent (Perlis, 1952, p. 172), and thus an n-by-n cyclic matrix has n linearly independent eigenvectors. It only remains to solve for them.

Let C be a cyclic matrix and denote the nonzero element on the i^{th} row of C as c_{1,t_i}. Let X by an eigenvector of C corresponding to eigenvalue λ. Then by definition

$$(\lambda I - C)X = 0, \qquad (63)$$

or, by the rules of matrix multiplication,

$$\begin{aligned} \lambda x_1 - c_{1,t_1} x_{t_1} &= 0 \\ \lambda x_2 - c_{2,t_2} x_{t_2} &= 0 \\ &\vdots \\ \lambda x_n - c_{n,t_n} x_{t_n} &= 0 \ . \end{aligned} \qquad (64)$$

Now the coefficient matrix in (63) has rank n-1; hence, the solution of the homogeneous system of linear equations contains a single arbitrary value. We set $x_1 = 1$. Then,

$$x_{t_1} = \frac{\lambda}{c_{1,t_1}} \qquad (65)$$

and furthermore

$$x_{t_{t_1}} = x_{t_1}\left(\frac{\lambda}{c_{t_1,t_{t_1}}}\right) . \qquad (66)$$

To eliminate the staircase subscripts we denote the i^{th}-fold image of 1 as s_i. Then, in general, the x_i are given by the recursion relationship

$$x_{s_i} = x_{s_{i-1}} \left(\frac{\lambda}{c_{s_{i-1},s_i}} \right) , \tag{67}$$

where $x_{s_0} = x_1 = 1$.

The subscripts $s_0, s_1, \ldots, s_{n-1}$ define the "cyclic order" of the cyclic matrix C. [Note that $s_0 = 1$.] For instance, the cyclic order of the cyclic matrix

$$\begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$$

is (1,4,2,3). Similarly, we define the "cyclic permutation" corresponding to the cyclic matrix C to be

$$\pi_c = \begin{pmatrix} 1,2,3, & \ldots,n \\ 1,s_1,s_2,\ldots,s_{n-1} \end{pmatrix},$$

whose corresponding permutation matrix is P_c. Then finally,

$$X = P_c \begin{pmatrix} \lambda^o/d_0 \\ \lambda/d_1 \\ \vdots \\ \lambda^{n-1}/d_{n-1} \end{pmatrix} , \tag{68}$$

where the d_i are given by the recursion

$$d_i = d_{i-1} \, c_{s_{i-1},s_i} \quad ; \quad i = 1,\ldots,n-1 \tag{69}$$

and $d_0 = 1$.

Letting D denote the diagonal matrix (d_{ii}), we can write the eigenvector X corresponding to the eigenvalue λ as

$$X = P_c D \begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{pmatrix} .$$

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of C, then the transform matrix is given by

$$S = P_c D \Lambda \; , \tag{70}$$

where

$$\Lambda = \begin{pmatrix} \lambda_1^o & \lambda_2^o & \cdots & \lambda_n^o \\ \lambda_1^1 & \lambda_2^1 & \cdots & \lambda_n^1 \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{pmatrix} . \tag{71}$$

Having treated the eigenvalues and eigenvectors of the cyclic submatrices of the cyclic norm form of Section 6, we

endeavor to show that the remaining eigenvalues of the cyclic normal form are all zero.

Recall that the cyclic normal form of a square row-elemental matrix of order n is

$$Q = \begin{pmatrix} C_1 & 0 \ldots & 0 & 0 \\ 0 & C_2 \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 \ldots & C_m & 0 \\ D_1 & D_2 \ldots & D_m & G \end{pmatrix}, \tag{72}$$

where $C_1, \ldots, C_m$ are cyclic submatrices and the submatrix G contains no cycles. For some positive integer r, the matrix G^r is identically zero. This fact follows from the argument that the elements of G^r given by the expression

$$\sum_{k_{r-1}=1}^{n} \cdots \sum_{k_2=1}^{n} \sum_{k_1=1}^{n} g_{i,k_1} g_{k_1,k_2} \cdots g_{k_{r-1},j} \tag{73}$$

in which the elements of the summand form a chain of length r. Since G contains no cycles such that the elements of G may be repeated in the summand, we can guarantee that G^r is zero simply by making r greater than the order of G: This follows from the fact that G is row elemental and therefore contains at most r nonzero elements.

For some power r of Q, say r equals the order of G, we can then say

$$Q^r = \begin{pmatrix} C_1^r & 0 \dots & 0 & 0 \\ 0 & C_2^r \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & C_m^r & 0 \\ * & * \dots & * & 0 \end{pmatrix}, \qquad (74)$$

where the submatrices (*) are of no particular concern here. What is of concern is that Q^r contains exactly r zero columns. The characteristic equation of Q^r therefore has the form

$$|\lambda I - Q^r| = \lambda^r g(\lambda) = 0 \qquad (75)$$

and hence has at least r zero eigenvalues. The following theorem of Perlis relates the eigenvalues of Q^r with those of Q.

Theorem 10

(Perlis, 1952, p. 168). Let A be a square matrix and let

$$g(x) = c(x)/d(x) ,$$

where c(x) and d(x) are polynomials and d(A) is non-singular. The if $\lambda_1,\dots,\lambda_n$ are the eigenvalues of A, $g(\lambda),\dots,g(\lambda_n)$ are the eigenvalues of g(A).

In our application, we let d(x) = 1 and $c(x) = x^r$. Hence, d(Q) = I and $g(Q) = Q^r$, whose eigenvalues are the eigenvalues of Q, raised to the r^{th} power. But Q has at least as many nonzero eigenvalues as it has cyclic elements, of which there are exactly n - r. Hence, Q^r has at least n - r nonzero eigenvalues by theorem 9. We therefore conclude that Q^r has

exactly r zero eigenvalues and finally, by the fact that the r^{th} root of zero is zero:

Theorem 11

If Q is a cyclic normal matrix having r noncyclic elements, Q has exactly r zero eigenvalues.

1.10 An Example

Consider a situation involving 3 Blue groups ($i = 1,2,3$) and 4 Red groups ($j = 1,2,3,4$). Blue's attrition matrix (the matrix of α_{ij}'s) and Red's attrition matrix are

$$[\alpha_{ij}] = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 1 & 4 & 2 \\ 1 & 1 & 2 & 1 \end{pmatrix}$$

and

$$[\beta_{ij}] = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \text{ respectively.}$$

The assignments of the Blue and Red groups are given by the 0,1 assignment matrices

$$E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then,

$$A = E \cdot [\alpha_{ij}] = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$B = H \cdot [\beta_{ji}] = \begin{pmatrix} 0 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

The attrition matrix C is then

$$C = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

We begin by finding the cyclic normal form. This is most easily done by listing the nonzero elements of C and checking each nonzero element individually to determine whether it is part of a cycle. In this case, we have

$$c(1,5) = 2,$$
$$c(2,6) = 4,$$
$$c(3,4) = 1,$$
$$c(4,2) = 4,$$
$$c(5,2) = 1,$$
$$c(6,2) = 1,$$
$$c(7,3) = 1 .$$

There is only one cycle in this example, and its elements are $c(2,6)$ and $c(6,2)$. The cyclic normal form of C is thus obtained by simultaneously interchanging rows and columns 1 and 6. Q is then obtained as

$$Q = PCP^{-1} = \left(\begin{array}{cc|ccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right),$$

which has been partitioned as a cyclic submatrix, the matrix, C and submatrices D and G. The matrix P is

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consider the cyclic submatrix first. We find two eigenvalues since C is of order 2, the eigenvalues being $\lambda_1 = 2$ and $\lambda_2 = -2$. As in Section 9, we may reduce C to diagonal form by pre- and post-multiplying C by S^{-1} and S, respectively, where $S = P_C D\Lambda$. In this case, P_C is the identity of order 2, D has diagonal elements

$$d_0 = 1$$

$$d_1 = 1$$

and

$$\Lambda = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} .$$

Thus,

$$S = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} , \quad S^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & -1/4 \end{pmatrix} .$$

The remaining S eigenvalues are all zero. The set of principal vectors of grade 1 related to the zero eigenvalues satisfy the relation

$$\bar{t}Q = 0 .$$

Inspection shows that there are only two such vectors (defined uniquely up to a scalar) and they are

$$\bar{t}_3 = (0\ 0\ 0\ 1\ -4\ 0\ 0)$$

and

$$\bar{t}_4 = (0\ 0\ 0\ 1\ 0\ -4\ 0) ,$$

the vectors $\bar{t}_1$ and $\bar{t}_2$ being the eigenvectors arising from the cyclic portion of Q.

Principal vectors of grade 2, i.e., satisfying

$$\bar{t}Q = \bar{t}_k \quad ,$$

where $\bar{t}_k$ is of grade 1, are now determined. Let $\bar{t}_5$ be the grade 2 principal vector which chains to $\bar{t}_3$. Then $\bar{t}_3$ satisfies

$$\bar{t}_5 Q = \bar{t}_3$$

and $\bar{t}_5$ is linearly independent of $\bar{t}_3$. The vector $\bar{t}_5$ is obtained by inspection to be (not uniquely)

$$\bar{t}_5 = (-2\ \ 0\ \ 1\ \ 0\ \ 1\ \ -1\ \ 0) \quad .$$

The vector $\bar{t}_6$ chains to $\bar{t}_4$ in a similar fashion, and is found to be (again, not completely determined)

$$\bar{t}_6\ (0\ \ 1\ \ 1\ \ 0\ \ 1\ \ -1\ \ 0) \quad .$$

One more principal vector remains and it must be of grade 3. The question is if it chains to $\bar{t}_5$ or $\bar{t}_6$. It cannot chain to $\bar{t}_5$ because there is no way to premultiply Q by $\bar{t}_7$ and obtain a nonzero element in the first column of the product (the -2 element). Therefore, again by inspection,

$$\bar{t}_7 = (1/2\ \ -1/4\ \ 0\ \ 0\ \ 0\ \ 1\ \ 1),$$

which completes the set of principal vectors.

The zero Jordan normal form of Q, therefore, can be written in the form

$$J^{\circ} = \left(\begin{array}{cc|cc|ccc} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) ,$$

where the matrix transformation T into J° is

$$T = (\bar{t}_1\ \bar{t}_2\ \bar{t}_3\ \bar{t}_5\ \bar{t}_6\ \bar{t}_7)' \quad ,$$

the prime indicating a transpose.

Written out in its entirety,

$$T = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -4 & 0 \\ -2 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & -1 & 0 \\ -\frac{1}{2} & -\frac{1}{4} & 0 & 0 & 0 & 1 & 1 \end{pmatrix} .$$

Finally, the matrix exponential e^{-tC} can be written in the closed form

$$e^{-tC} = R^{-1}F(-t)R$$

or

$$e^{-tC} = R^{-1}\left(\begin{array}{cc|cc|ccc}
e^{-2t} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{2t} & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & -t & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & -t & t^2/2 \\
0 & 0 & 0 & 0 & 0 & 1 & -t \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right),$$

where $R = PT$. Given initial force component vectors $\overline{M}$ and $\overline{N}$, we then obtain solutions to the heterogeneous-force equations illustrated above as

$$(\overline{m}\ \overline{n}) = (\overline{M}\ \overline{N})R^{-1}F(-t)R \quad .$$

1.11 References

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Appendix D, 1, 1

PROOF OF THEOREM[1] 1

Stanley Sternberg

The method of successive approximations is employed to establish the existance of a solution of the matrix differential equation

$$\frac{dX}{dt} = XA(t), \quad X(0) = I . \tag{1}$$

In place of (1), we consider the integral equation

$$X = I + \int_0^t XA(s)ds . \tag{2}$$

Now, define the sequence of matrices $\{X_n\}$ as follows:

$$\begin{aligned} X_o &= I \\ X_{n+1} &= I + \int_0^t X_n A(s)ds \\ &\qquad n = 0,1,\ldots. \end{aligned} \tag{3}$$

Then we have

$$\begin{aligned} X_{n+1} - X_n &= \int_0^t (X_n - X_{n-1})A(s)ds \\ &\qquad n = 1,2,\ldots. \end{aligned} \tag{4}$$

Let

$$m = \max_{0 \leq s \leq t} \|A(s)\|$$

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[1] This proof is given in Bellman, R., *Introduction to Matrix Analysis,* New York: McGraw-Hill Book Company, 1960.

be the maximum norm of A(s) in (0, t), where the norm is defined by

$$\|A\| = \sum_{i=1}^{N}\sum_{j=1}^{N} |a_{ij}| \ . \tag{5}$$

Using (4), we obtain

$$\begin{aligned}\|X_{n+1} - X_n\| &= \left\|\int_0^t (X_n - X_{n-1})A(s)ds\right\| \\ &\leq \int_0^t \|X_n - X_{n-1}\|\,\|A(s)\|ds \\ &\leq m\int_0^t \|X_n - X_{n-1}\|ds \end{aligned} \tag{6}$$

for $0 \leq t \leq t_1$. Since, in this same interval,

$$\|X_1 - X_o\| \leq \int_0^t \|A(s)\|ds \leq mt \tag{7}$$

we have inductively from (6),

$$\|X_{n+1} - X_n\| \leq \frac{m^{n+1}t^{n+1}}{(n+1)!} \qquad \text{for } 0 \leq t \leq t_1 \ . \tag{8}$$

Hence, the series $\sum_{n=0}^{\infty} (X_{n+1} - X_n)$ converges uniformly for $0 \leq t \leq t_1$. Consequently, X_n converges uniformly to a matrix X(t) which satifies (2), and thus (1).

Since, by assumption, A(t) is continuous for $t \geq 0$, we may take t, arbitrarily large. We thus obtain a solution valid for $t \geq 0$.

It is easily verified that $\bar{x} = \bar{q}X(t)$ is a solution of

$$\frac{d\bar{x}}{dt} = \bar{x}A(t) \qquad \bar{x}(0) = \bar{q} \, , \tag{9}$$

satisfying the required initial condition. We now establish the uniqueness of this solution.

Let Y be another solution of (1). Then Y satisfies (2), and thus we have the relation

$$X - Y = \int_0^t [X(s) - Y(s)]A(s)ds \quad . \tag{10}$$

Hence

$$\|X - Y\| \leq \int_0^t \|X(s) - Y(s)\| \, \|A(s)\| \, ds \; . \tag{11}$$

Since y is differentiable, hence continuous, define

$$m_1 = \max_{0 \leq t \leq t_1} \|X - Y\| \, . \tag{12}$$

From (12), we obtain

$$\|X - Y\| \leq m_1 \int_0^t \|A(s)\| \, ds \qquad 0 \leq t \leq t_1 \; . \tag{13}$$

Using this bound in (11), we obtain

$$\|X - Y\| \leq m_1 \int_0^t \left(\int_0^s \|A(s_1)\| \, ds_1 \right) \|A(s)\| \, ds$$

$$\leq \frac{m_1\left(\int_0^t \|A(s)\|\, ds\right)^2}{2} \quad . \tag{14}$$

Integrating, we obtain

$$\|X - Y\| \leq \frac{m_1^n\left(\int_0^t \|A(s)\|\, ds\right)^{n+1}}{n+1} \quad . \tag{15}$$

Letting $n \to \infty$, we see that $\|X - Y\| \leq 0$. Hence $X \equiv Y$.

Having obtained the matrix X, it is easy to see that $\bar{q}X(t)$ is a solution of (9). Since the uniqueness of the solution of (9) is readily established by means of the same argument as above, it is easy to see that $\bar{q}X(t)$ is *the* solution.

Appendix D, 1, 2

PROOF OF CONVERGENCE OF THE MATRIX EXPONENTIAL[1]

Stanley Sternberg

The matrix exponential is defined by means of the infinite series

$$e^{At} = I + At + \cdots + \frac{A^n t^n}{n!} + \cdots . \quad (1)$$

Theorem

The matrix series defined above exists for all A for any fixed value of t, and for all t for any fixed A. It converges uniformly in any finite region of the complex t plane.

Proof

We have

$$\frac{\|A^n t^n\|}{n!} \leq \frac{\|A\|^n |t|^n}{n!} \quad (2)$$

Since $\|A\|^n |t|^n/n!$ is a term in the series expansion of $e^{\|A\| \, |t|}$, we see that the series of (1) is dominated by a uniformly convergent series, and hence is itself uniformly convergent in any finite region of the t plane.

[1] This proof is given in Bellman, R., *Introduction to Matrix Analysis*, New York: McGraw-Hill Book Company, 1960.

Chapter 2

ALLOCATION STRATEGIES

Stanley Sternberg

2.1 Introduction

In the preceding chapter, a method of obtaining force size versus time solutions for the constant-coefficient, heterogeneous-force model was presented. The method was general in that it allowed for various combinations of weapon assignments, but special in that all weapons of a single class were to be assigned to a single opposition weapon class, the so-called 0,1 assignments.

It is clear that the particular time solution obtained from a given set of initial conditions depends upon the choice of weapon assignments. A set of rules which govern how a particular weapon class is going to be assigned constitutes an assignment strategy. These rules are most commonly determined by attempting to synthesize how weapons would be employed in a real combat situation, but, there being no hard and fast rules in the real world, the definition of assignment strategies in the model becomes a subjective judgment on the part of the modeler.

It is generally true that no two assignment strategies yield the same results. Thus, the performance of a particular weapon class in a combat model depends implicitly on the judgment of the user of how that weapon, and all other

weapons in the combat model, are "best" employed. Since analysis of weapon systems and force structures using the heterogeneous-force model depends on the assignments employed, it is of interest to determine good allocation strategies for each force considered, i.e., let each force use its capabilities to best advantage.

This chapter attempts to generate a unique "best" set of assignment rules, based solely upon a set of weapon characteristics defined as initial conditions on the model. The approach leans heavily on the theory of differential games as developed by Isaacs (1965), and the reader is referred there for a more complete discussion of some of the concepts discussed herein.

2.2 The Heterogeneous-Force Differential Equations

The equations describing the attrition process in the heterogeneous case, as previously discussed, are

$$\frac{dm_i}{dt} = - \sum_{j=1}^{J} h_{ji}\beta_{ji}n_j \; ; \qquad i = 1,\ldots,I \tag{1}$$

and

$$\frac{dn_j}{dt} = - \sum_{i=1}^{I} e_{ij}\alpha_{ij}m_i \; ; \qquad j = 1,\ldots,J \; , \tag{2}$$

where α_{ij} and β_{ji} are the Blue and Red attrition rates, respectively; e_{ij} is the fraction of the i^{th} Blue group assigned to the j^{th} Red group; and h_{ji} is the fraction of the j^{th} Red group assigned to the i^{th} Blue group. Naturally,

$$\sum_{j=1}^{J} e_{ij} \leq 1 \text{ and } e_{ij} \geq 0 \ ; \quad i = 1,\ldots,I \tag{3}$$

and

$$\sum_{i=1}^{I} h_{ji} \leq 1 \text{ and } h_{ji} \geq 0 \ ; \quad j = 1,\ldots,J \ . \tag{4}$$

The course of the battle is completely described by the I+J simultaneous differential equations together with the initial numbers of each Red and Blue groups and the choices of the weapon assignments e_{ij} and h_{ji} throughout the battle. It is assumed that weapon assignments can be modified at any time during the course of the battle without penalty and that both Red and Blue have full knowledge of the numbers of all weapon types at all times during the conflict. A plan for choosing and possibly modifying e_{ij} and h_{ji} in accordance with constraints (3) a. (4) is said to constitute the "Blue or Red assignment strategy."

2.2 *Introductory Concepts*

We define two (I+J) dimensional row vectors $\bar{z}$ and $d\bar{z}/dt$ by

$$\bar{z} = (m_1 \cdots m_I n_1 \cdots n_J)$$

and

$$d\bar{z}/dt = \left(\frac{dm_1}{dt} \cdots \frac{dm_I}{dt}\, \frac{dn_1}{dt} \cdots \frac{dn_J}{dt}\right)$$

and denote the k^{th} element of each as z_k and dz_k/dt, respectively. Using this notation, the heterogeneous differential combat model can be denoted as

$$d\bar{z}/dt = \bar{f}(\bar{z},E,H), \qquad (5)$$

where $\bar{f}$ is a row-vector function whose k^{th} element is $f_k(\bar{z},E,H) = dz_k/dt$ and E and H are the matrices

$$E = [e_{ij}] \text{ and } H = [h_{ji}] \quad .$$

Since the elements of $\bar{z}$ must be non-negative, we might think of a point $\bar{z}$ to be in motion in the positive octant E^+ of Euclidean (I+J)-space, its motion governed by equation 5. The matrices E and H are the "controls" exercised by Blue and Red, respectively, for influencing the motion of $\bar{z}$. We will speak of a particular position of $\bar{z}$ in E^+ as the "state" of the battle and the path that $\bar{z}$ travels in E^+ as the "trajectory" of the battle.

We define a surface C in E^+, called the "terminal surface," such that when $\bar{z}$ reaches C, the battle is over. One example of a terminal surface might be all those points in E^+ such that Red or Blue or both Red and Blue are annihilated.

Or we might choose our terminal surface on the condition that either Red or Blue will retreat from the battle when his casualties are greater than 90 percent and so on. We shall also require a stopping rule in case $\bar{z}$ never reaches C; that is, we select some value (T) of time and decree that the battle is over if T elapses.

We assume that it is the dual intention of each combatant to inflict maximum casualties on his opponent while minimizing his own, subject to being involved in the engagement. The success of each opponent in pursuing his intention is measured by a payoff function. The payoff function is assumed to be of the terminal type, that is, if $\bar{z}_f$ is the end point of a trajectory terminating on C, or if the upper time limit T is reached, then the payoff is $P(\bar{z}_f)$. We will adopt the convention that the payoff is made to the Red force. Thus, the Red force seeks to exercise his control on the trajectory of $\bar{z}$ such that the payoff will be maximized when $\bar{z}$ reaches C or $t = T$. Similarly, Blue determines his control on the motion of $\bar{z}$ in such a way that the payoff is minimized.

A rule for choosing a set of apportionments (controls) E or H for all possible positions of $\bar{z}$ constitutes a "strategy," i.e., a determination of the functions $E(\bar{z})$ and $H(\bar{z})$. Both players seek "optimal strategies" $E^*(\bar{z})$ and $H^*(\bar{z})$, such that for every other strategy $E(\bar{z})$ and $H(\bar{z})$, the payoff P resulting from the application of E^* and H^* starting from the point $\bar{z}$

satisfies

$$P[\bar{z},E^*,H] \leq P[\bar{z},E^*,H^*] \leq P[\bar{z},E,H^*)] \quad . \tag{6}$$

Equation 6 states that if the battle is in state $\bar{z}$ and Blue employs an optimal strategy E^*, then Red will maximize the payoff by employing strategy H^* rather than some other strategy H. Similarly, with Red using H^*, Blue will achieve the minimum payoff by employing E^* rather than some other strategy E. Such a couple, E^*, H^* constitutes a "saddle point." The corresponding value of the payoff which results when E^* and H^* are employed throughout the battle,

$$U(\bar{z}) = P[\bar{z},E^*,H^*] \quad , \tag{7}$$

is simply called the "value" of the conflict, and, if it exists, it is unique. The optimal strategies E^* and H^* may not be unique, but if there are more than one, they are equivalent in that they yield equivalent payoffs.

2.4 The Main Equation

Suppose a saddle point exists and suppose both opponents employ their optimal strategies. Then the expected payoff of the battle is the value U of the conflict and its magnitude depends only upon the starting point $\bar{z}_o$. We assume that the trajectory $z(t)$ in E^+ is piecewise differentiable. Then, over the smooth portions of $\bar{z}(t)$ the total derivative of $U(\bar{z})$ is zero, i.e.,

$$\sum_{k=1}^{I+J} \frac{\delta U}{\delta z_k} \cdot \frac{dz_k}{dt} = 0 \ . \qquad (8)$$

Since $dz_k/dt = f_k(\bar{z},E^*,H^*)$ on the optimal trajectory, then denoting

$$u_k \text{ for } \frac{\delta U}{\delta z_k} \ ,$$

equation 8 becomes

$$\sum_{k=1}^{I+J} u_k f_k(\bar{z},E^*,H^*) = 0 \qquad (9)$$

or

$$\min_E \max_H \sum_{k=1}^{I+J} u_k f_k(\bar{z},E,H) = 0 \quad . \qquad (10)$$

Note that as long as Blue and Red hold to the optimal strategies E* and H*, then the value remains constant over the trajectory. If, however, for example, Blue departs from E* to some nonoptimal E, i.e., he fails to minimize the left-hand side of equation 10, then the rate of change of U is positive,

$$\sum_{k-1}^{I+J} u_k f_k(\bar{z},E,H^*) > 0 \quad , \qquad (11)$$

and the value of the conflict shifts in favor of Red. It is conceivable, of course, that Red could shift from H* to take

further advantage of Blue's nonoptimal play, but then, in doing so, he leaves himself vulnerable to further changes in Blue's strategy. The minimax strategy can be considered, therefore, to be somewhat conservative, but however, as long as Red holds to H*, then no matter what strategy Blue employs, red will do no worse than $P(\bar{z}_o, E^*, H^*)$, and if Blue departs from E*, he will do better. Similar remarks, of course, apply to Blue.

Equation 9 is referred to as the "main equation."

2.5 *The Determination of Optimal Assignment Strategies*

We begin with a change in notation. Recalling that the elements of $\bar{z}$ are the m_i and n_j, then we write the partial derivatives of U in the following form:

$$\frac{\delta U}{\delta m_i} = v_i ; \quad i = 1, \ldots, I \tag{12}$$

and

$$\frac{\delta U}{\delta n_j} = w_j ; \quad j = 1, \ldots, J \quad . \tag{13}$$

Substituting equations 1 and 2 into equation 10 yields

$$\min_E \max_H \left[\sum_{i=1}^{I} v_i \left(- \sum_{j=1}^{J} h_{ji} \beta_{ji} n_j \right) + \sum_{j=1}^{J} w_j \left(- \sum_{i=1}^{I} e_{ij} \alpha_{ij} m_i \right) \right] = 0 \tag{14}$$

or, multiplying both sides of (14) by -1 and reversing the summation order,

$$\max_{E}\ \min_{H}\left[\sum_{j=1}^{J} n_j \sum_{i=1}^{I} v_i h_{ji} \beta_{ji} + \sum_{i=1}^{I} m_i \sum_{j=1}^{J} w_j e_{ij} \alpha_{ij}\right] = 0 \quad (15)$$

A maximum over E of the bracketed expression satisfying the constraints of equation 3 is obtained by setting

$$e^{*}_{ij} = \begin{cases} 1 \text{ if } w_j \alpha_{ij} = \max\limits_{k=1,\ldots,J}\left[w_k \alpha_{ik}\right] \\ \quad \text{and } w_j > 0 \\ 0 \text{ otherwise} \end{cases} ;\ \text{for } i = 1,\ldots,I\ . \quad (16)$$

The minimum over H satisfying (4) is

$$h^{*}_{ji} = \begin{cases} 1 \text{ if } v_i \beta_{ji} = \min\limits_{k=1,\ldots,I}\left[v_k \beta_{jk}\right] \\ \quad \text{and } v_i < 0 \\ 0 \text{ otherwise} \end{cases} ;\ \text{for } j = 1,\ldots,J\ . \quad (17)$$

In the instance of a tie of the maximum $w_j \alpha_{ij}$ or minimum $v_i \beta_{ji}$ any apportionment of weapon type i or j between the tying target types is optimal. Also, we have restricted the trajectory of $\bar{z}$ to E^{+}, so any assignment to an empty weapon class is not permitted (or we might redefine α_{ij} and β_{ji} such that α_{ij} and β_{ji} are zero when n_j or m_i are zero, respectively.)

Then,

$$h^*_{ji} = 0 \text{ if } m_i = 0 \text{ ; for } j = 1,\ldots,J \tag{18}$$

and

$$e^*_{ij} = 0 \text{ if } n_j = 0 \text{ ; for } i = 1,\ldots,I \quad . \tag{19}$$

The optimal assignment strategies of equations 16 and 17 state that all weapons of type i or type j should be assigned to a single opposition weapon class. The choice of which weapon class is to be fired upon is independent of the number of weapons in the firing class. The class to be fired upon is selected by determining the maximum attrition rates on the marginal utilities of the opposing weapon classes and not directly by the numbers of weapons in the opposing weapon classes.

2.6 The Path Equations

In Section 2.5 it was shown that the optimal assignment strategies depend on the values of the partial derivatives of the value function U. Therefore, what is needed is a method of determining the v_i and w_j at each point in the battle trajectory.

We begin with equation 9,

$$\sum_{k=1}^{I+J} u_k f_k(\tilde{z},E^*,H^*) = 0 \quad .$$

Differentiating with respect to each z_ℓ, ($\ell = 1,2,\ldots,I+J$),

$$\frac{\delta}{\delta z_\ell}\left[\sum_k u_k f_k\right] = \sum_k \frac{\delta u_k}{\delta z_\ell} f_k + \sum_k u_k \frac{\delta f_k}{\delta z_\ell} = 0 \ . \tag{20}$$

But

$$\frac{\delta u_k}{\delta z_\ell} = \frac{\delta^2 U}{\delta z_k \delta z_\ell} = \frac{\delta u_\ell}{\delta z_k} \tag{21}$$

and since $f_k = \frac{dz_k}{dt}$, equation 20 becomes

$$\sum_k \frac{\delta u_\ell}{\delta z_k} \cdot \frac{dz_k}{dt} + \sum_k u_k \frac{\delta f_k}{\delta z_\ell} = 0 \tag{22}$$

or

$$\frac{du_\ell}{dt} = - \sum_{k=1}^{I+J} u_k f_{k\ell}(\bar{z},E^*,H^*); \text{ for } \ell = 1,\ldots,I+J \ , \tag{23}$$

where

$$f_{k\ell}(\bar{z},E^*,H^*) \text{ denotes } \frac{\delta f_k(\bar{z},E^*,H^*)}{\delta z_\ell} \ .$$

Equation 23, when expanded into its Blue and Red weapon components, becomes

$$\frac{du_\ell}{dt} = -\sum_{k=1}^{I} u_k \frac{\delta f_k}{\delta z_\ell} - \sum_{k=I+1}^{I+J} u_k \frac{\delta f_k}{\delta z_\ell} \; ; \text{ for } \ell = 1,\ldots,I \qquad (24)$$

and

$$\frac{du_\ell}{dt} = -\sum_{k=1}^{I} u_k \frac{\delta f_k}{\delta z_\ell} - \sum_{k=I+1}^{I+J} u_k \frac{\delta f_k}{\delta z_\ell} \; ; \text{ for } \ell = I+1,\ldots,I+J \qquad (25)$$

But from equation (1) it is seen that for $k = 1,\ldots,I$

$$\frac{\delta f_k}{\delta z_\ell} = 0 \; ; \text{ for } \ell = 1,\ldots,I \qquad (26)$$

and

$$\frac{\delta f_k}{\delta z_\ell} = -h^*_{k,\ell-I}\beta_{k,\ell-I} \; ; \text{ for } \ell = I+1,\ldots,I+J \; . \qquad (27)$$

Similarly, from equation 2 , for $k = I+1,\ldots,I+J$,

$$\frac{\delta f_k}{\delta z_\ell} = -e^*_{k-I,\ell}\alpha_{k-I,\ell} \; ; \text{ for } \ell = 1,\ldots,I \qquad (28)$$

and

$$\frac{\delta f_k}{\delta z_\ell} = 0 \; ; \text{ for } \ell = I+1,\ldots,I+J \; . \qquad (29)$$

Equations 24 and 25 then become

$$\frac{du_\ell}{dt} = \sum_{k=I+1}^{I+J} u_k e^*_{k-I,\ell} \alpha_{k-I,\ell}; \quad \text{for } \ell = 1,\ldots,I \tag{30}$$

$$\frac{du_\ell}{dt} = \sum_{k=1}^{I} u_k h^*_{k,\ell-I} \beta_{k,\ell-I} ; \quad \text{for } \ell = I+1,\ldots,I+J . \tag{31}$$

Now letting $i = \ell$ and $j = k - I$ in (30) and letting $j = \ell - I$ and $i = k$ in (31),

$$\frac{dv_i}{dt} = \sum_{j=1}^{J} e^*_{ji} \alpha_{ji} w_j; \quad \text{for } i = 1,\ldots,I \tag{32}$$

$$\frac{dw_j}{dt} = \sum_{i=1}^{I} h^*_{ij} \beta_{ij} v_i ; \quad \text{for } j = 1,\ldots,J . \tag{33}$$

Equations 32 and 33, together with equations 1 and 2, define a system of linear differential equations which are referred to as the "path equations" of the combat process. Equations 1 and 2 describe the process of attrition on the number of weapons in a weapon group, while equations 32 and 33 describe the process of attrition on the marginal utilities of the weapons within a weapon group. The numbers and utilities of weapons are related in an interesting fashion in the next section.

2.7 *A State Equation*

We reintroduce the row-vector notation

$$\bar{m} = (m_1,\ldots,m_I) \text{ and } \bar{n} = (n_1,\ldots,n_J)$$

while defining $\bar{v}$ and $\bar{w}$ as

$$\bar{v} = (v_1,\ldots,v_I) \text{ and } \bar{w} = (w_1,\ldots,w_J) \quad .$$

Attrition matrices A* and B* are

$$A^* = [a^*_{ij}] = [e^*_{ij}\alpha_{ij}]$$

and

$$B^* = [b^*_{ji}] = [h^*_{ji}\beta_{ji}] \quad .$$

The heterogeneous-path equations can now be written in the compact notation

$$d\bar{m}/dt = -\bar{n}B^* \tag{34}$$

$$d\bar{n}/dt = -\bar{m}A^* \tag{35}$$

$$d\bar{v}/dt = \bar{w}A^{*T} \tag{36}$$

$$d\bar{w}/dt = \bar{v}B^{*T} \tag{37}$$

or, in a still more compact form by adding to our previous row vector $\bar{z}$ and a row vector $\bar{u}$,

$$\bar{u} = (\bar{v},\bar{w})$$

such that equations 34 through 37 become

$$d\bar{z}/dt = -\bar{z}C^* \tag{38}$$

and

$$d\bar{u}/dt = \bar{u}C^{*T} , \tag{39}$$

where the matrix C* of order I+J is

$$C^* = \begin{pmatrix} 0 & A^* \\ B^* & 0 \end{pmatrix} .$$

The solution of equations 38 and 39 have been discussed in detail in Chapter 1. They are given by

$$\bar{z} = \bar{z}_o\ e^{-tC^*} \tag{40}$$

and

$$\bar{u} = \bar{u}_o\ e^{tC^{*T}} , \tag{41}$$

where the matrix exponential of the form e^{tA} is

$$e^{tA} = I + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots . \tag{42}$$

Noting that

$$(A^T)^n = (A^n)^T \tag{43}$$

so that

$$\begin{aligned} e^{tA^T} &= I^T + (tA)^T + \left(\frac{t^2A^2}{2!}\right)^T + \left(\frac{t^3A^3}{3!}\right) + \cdots \\ &= (e^{tA})^T , \end{aligned} \tag{44}$$

equation 41 can be rewritten as

$$\bar{u} = \bar{u}_o (e^{tC^*})^T \tag{45}$$

or

$$\bar{u}^T = e^{tC^*} \bar{u}_o^T \quad . \tag{46}$$

Our state equation results from forming the product

$$\bar{z}\,\bar{u}^T = \bar{z}_o\, e^{-tC^*}\, e^{tC^*}\, \bar{u}_o^T \tag{47}$$

or finally

$$\bar{z}\,\bar{u}^T = \bar{z}_o\,\bar{u}_o^T \; . \tag{48}$$

The product $\bar{z}\,\bar{u}^T$ is a scalar quantity. Equation 48 states that if both opponents exercise their optimal strategies, then $\bar{z}\,\bar{u}^T$ is invariant throughout the conflict. Suppose that $\bar{z}_f$ and $\bar{u}_f$ are the vectors $\bar{z}$ and $\bar{u}$ evaluated at a terminal surface. Then, $\bar{z}_f \bar{u}_f^T$ is the terminal payoff. Our state equation is then a restatement of the fact that if both sides employ optimal assignment strategies, the expected payoff is constant throughout the game. But the value of the payoff when optimal assignments are used is the definition of the value of the game, U. Hence,

$$U(\bar{z}, E^*, H^*) = \bar{z}\,\bar{u}^T \quad . \tag{49}$$

The total derivative of U is zero. Rewriting equation 49 in terms of the components of $\bar{z}$ and $\bar{u}$, i.e.,

$$U = \sum_{k=1}^{I+J} z_k u_k \tag{50}$$

and differentiating with respect to time gives

$$\sum_{k=1}^{I+J} z_k \frac{du_k}{dt} + \sum_{k=1}^{I+J} \frac{dz_k}{dt} u_k = 0 \quad . \tag{51}$$

But the second term of (51) is identically zero as it is the main equation. Therefore,

$$\sum_{k=1}^{I+J} z_k \frac{du_k}{dt} = 0 \tag{52}$$

is an alternate form of the main equation in the case of our heterogeneous-force combat model.

2.8 Three-Dimensional Assignment Example

The simplest example of a heterogeneous differential combat formulation involving the determination of an optimal fire allocation policy occurs in the two-on-one battle. There Blue, who possesses a single weapon type in one group whose quantity is m, assigns a fraction e of those weapons to fire upon Red's group-1 weapon whose quantity is n_1, the remainder of Blue's force being assigned to Red's group-2 weapons. The differential equations describing the attrition rate of each weapon group are

$$\frac{dm}{dt} = -\beta_1 n_1 - \beta_2 n_2$$

$$\frac{dn_1}{dt} = -e\alpha_1 m \qquad (53)$$

$$\frac{dn_2}{dt} = -(1-e)\,\alpha_2 m \quad ,$$

where the attrition coefficients are all nonzero. The battle terminates at time t_f, at which point all the weapons on one side or the other have been annihilated. (Conceivably, this event could take infinite time). At termination, the payoff to red is

$$\text{Payoff} = d_1 n_1(t_f) + d_2 n_2(t_f) - cm(t_f). \qquad (54)$$

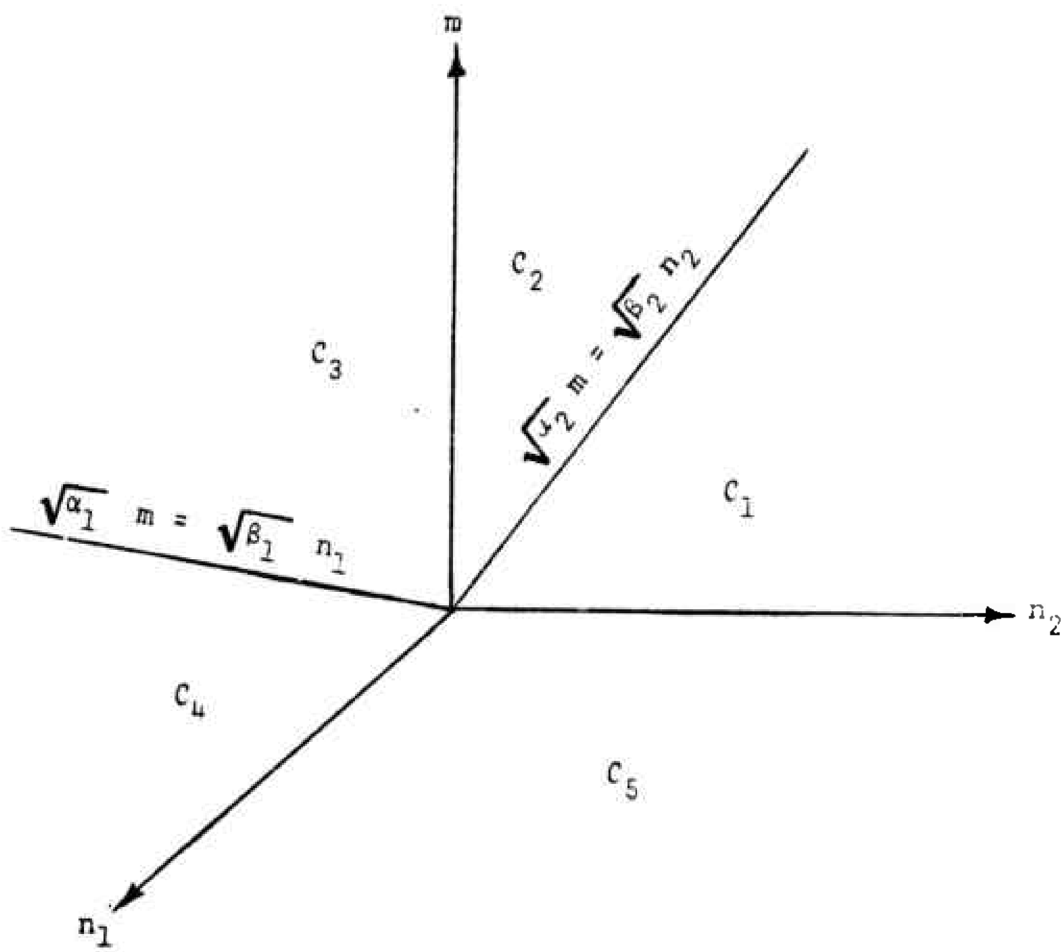
Blue's strategy is then to select e through the course of the battle as to minimize the eventual payoff.

The terminal surface C for our three-dimensional game is the boundary of the first octant in 3-space. We label these bounding planes C_1, C_3, and C_5 as either n_1, n_2, or m are zero, respectively. We further partition C_1 into C_1 and C_2 and C_3 into C_3 and C_4 in accordance with the eventual winner of the game.

From that point in time at which any one weapon type is annihilated, the eventual outcome is completely determined. If m is reduced to zero first, the battle terminates. If either n_1 or n_2 is annihilated, a simple homogeneous-force, constant attrition-coefficient conflict ensues. The eventual out- [illegible] is [illegible] obtained from the square law [illegible] ...

$$\alpha_i[m^2 - m(t_f)^2] = \beta_j[n_j^2 - n_j(t_f)^2]; \quad \text{for } j = 1,2 \qquad (55)$$

by setting either $m(t_f)$ or $n_i(t_f)$ to zero. The boundaries of Euclidean 3-space can then be partitioned as shown in Figure 1 with the eventual payoffs, H, as determined by equations 54 and 55, shown in equation 56.



PARTITIONING THE SPACE

Figure 1

$$H = \frac{d_2}{\sqrt{\beta_2}} \sqrt{\beta_2 n_2^{\,2} - \alpha_2 m^2} \qquad \text{on } C_1$$

$$H = -\frac{c}{\sqrt{\alpha_2}} \sqrt{\alpha_2 m^2 - \beta_2 n_2^{\,2}} \qquad \text{on } C_2$$

$$H = -\frac{c}{\sqrt{\alpha_1}} \sqrt{\alpha_1 m^2 - \beta_1 n_1^{\,2}} \qquad \text{on } C_3 \tag{56}$$

$$H = \frac{d_1}{\sqrt{\beta_1}} \sqrt{\beta_1 m_1^{\,2} - \alpha_1 n^2} \qquad \text{on } C_4$$

$$H = d_1 n_1 + d_2 n_2 \qquad \text{on } C_5 .$$

The subscript labeling of the Red weapon groups is completely arbitrary, and so we choose to assume the condition that

$$\alpha_1 \beta_1 \geq \alpha_2 \beta_2 \ . \tag{57}$$

Furthermore, our eventual results will depend very heavily on two cases, these being

$$\begin{aligned} &\text{Case I:} \quad \alpha_1 d_1 > \alpha_2 d_2 \\ &\text{Case II:} \quad \alpha_1 d_1 < \alpha_2 d_2 \ . \end{aligned} \tag{58}$$

In Case I, our result will show that it is optimal for Blue to allocate all its firepower at Red group-1 whenever $n_1 > 0$. In Case II, the greater value of the Red group-2 weapon in

the payoff makes it optimal for Blue, at some point in the battle, to assign all his firepower at Red group-2.

The main equation for this conflict is

$$\min_{e} \left[v \frac{dm}{dt} + w_1 \frac{dn_1}{dt} + w_2 \frac{dn_2}{dt} \right] = 0 , \tag{59}$$

where v, w_1 and w_2 denote $\frac{\partial U}{\partial m}$, $\frac{\partial U}{\partial n_1}$, $\frac{\partial U}{\partial n_2}$, respectively, and U is the value function. Substituting into the main equation the appropriate derivatives yields

$$\min_{e} \left[-em(\alpha_1 w_1 - \alpha_2 w_2) - \alpha_2 m w_2 - (\beta_1 n_1 + \beta_2 n_2)v \right] = 0. \tag{60}$$

Letting S denote the quantity $(\alpha_1 w_1 - \alpha_2 w_2)$, it is seen that the minimizing value of e, namely e*, is

$$e^* = \begin{cases} 1 & \text{whenever} \quad S > 0 \\ 0 & \text{whenever} \quad S < 0 \end{cases} \tag{61}$$

and indeterminant in the case of S = 0. Obviously, w_1 and w_2 are always positive, meaning the addition of an extra group-1 or group-2 weapon at any time during the battle will result in a larger eventual payoff to Red. Thus, Blue's optimal allocation tactic is simply to assign all of his firepower at Red group-i if $\alpha_i w_i$ is greater than $\alpha_j w_j$, $i \neq j$.

The retrogressive path equations to this game are

$$\frac{dm}{d\tau} = \beta_1 n_1 + \beta_2 n_2$$

$$\frac{dn_1}{d\tau} = e^* \alpha_1 m$$

$$\frac{dn_2}{d\tau} = (1-e^*)\alpha_2 m$$

$$\frac{dv}{d\tau} = - e^*\alpha_1 w_1 - (1-e^*)\alpha_2 w_2 \tag{62}$$

$$\frac{dw_1}{d\tau} = - \beta_1 v$$

$$\frac{dw_2}{d\tau} = - \beta_2 v .$$

Thus,

$$\frac{d}{d\tau} (\alpha_1 w_1 - \alpha_2 w_2) = - (\alpha_1 \beta_1 - \alpha_2 \beta_2) v \quad . \tag{63}$$

But v is always negative, meaning the addition of a single blue weapon at any time during the battle reduces the eventual payoff to Red. Thus, $\frac{dS}{d\tau} > 0$ because $\alpha_1\beta_1 > \alpha_2\beta_2$. Hence, S is strictly increasing with increasing τ, except for possible jumps at transition surfaces.

We begin our analysis by examining trajectories terminating on C_5. We parameterize C_5 as $m = 0$, $n_1 = s_1$, $n_2 = s_2$. The value of the conflict on C_5 is

$$U = H = d_1 s_1 + d_2 s_2$$

so that on C_5 we have

$$w_1 = \frac{\delta U}{\delta n_1} = \frac{\delta U}{\delta s_1} \cdot \frac{\delta s_1}{\delta n_1} + \frac{\delta U}{\delta s_2} \cdot \frac{\delta s_2}{\delta n_1} = \frac{\delta U}{\delta s_1} = d_1$$

and

$$w_2 = \frac{\delta U}{\delta n_2} = \frac{\delta U}{\delta s_1} \cdot \frac{\delta s_1}{\delta n_2} + \frac{\delta U}{\delta s_2} \cdot \frac{\delta s_2}{\delta n_2} = \frac{\delta U}{\delta s_2} = d_2 .$$

The main equation must hold on C_5. Substituting $m = 0$ into equation 60 yields $v = 0$.

Now on C_5, the quantity S depends upon Case I and Case II.

Case I

Since $\alpha_1 w_1 > \alpha_2 w_2$ on C_5, then $S > 0$ and $e^* = 1$ in the neighborhood of C_5. The retrogressive path equations of (62) become

$$\begin{aligned} \dot{m} &= \beta_1 n_1 + \beta_2 n_2 & \dot{v} &= -\alpha_1 w_1 \\ \dot{n}_1 &= \alpha_1 m & \dot{w}_1 &= -\beta_1 v \qquad (64) \\ \dot{n}_2 &= 0 & \dot{w}_2 &= -\beta_2 v , \end{aligned}$$

where $(\dot{\ })$ denotes $\frac{d}{d\tau}(\ )$. Since S is strictly increasing with increasing τ, we are insured that $e^* = 1$ holds everywhere along

these paths. The solutions of (64) with initial conditions taken on C_5 can be obtained using the method of Chapter 1.

$$m = \frac{\beta_1 s_1 + \beta_2 s_2}{\sqrt{\alpha_1\beta_1}} \sinh\left(\sqrt{\alpha_1\beta_1}\,\tau\right)$$

$$n_1 = \frac{\beta_1 s_1 + \beta_2 s_2}{\beta_1} \cosh\left(\sqrt{\alpha_1\beta_1}\,\tau\right) - \frac{\beta_2 s_2}{\beta_1}$$

$$n_2 = s_2 \tag{65}$$

$$v = -\frac{d_1\sqrt{\alpha_1\beta_1}}{\beta_1} \sinh\left(\sqrt{\alpha_1\beta_1}\,\tau\right)$$

$$w_1 = d_1 \cosh\left(\sqrt{\alpha_1\beta_1}\,\tau\right)$$

$$w_2 = \frac{d_1\beta_2}{\beta_1}\left[\cosh\left(\sqrt{\alpha_1\beta_1}\,\tau\right) - 1\right] + d_2.$$

We now examine the space of initial conditions (at time $t = 0$) whose trajectories terminate on C_5. To do this we will establish the boundaries of this space.

Substituting $s_2 = 0$ into equations 65 yields those paths terminating on the n_2-axis. These paths lie in the m,n_1 plane since $s_2 = 0$ dictates that $n_2 = 0$. The equations of these paths are

$$m = \sqrt{\frac{\beta_1}{\alpha_1}}\; s_1 \;\sinh\left(\sqrt{\alpha_1\beta_1}\;\tau\right)$$

(66)

$$n_1 = s_1 \cosh\left(\sqrt{\alpha_1\beta_1}\;\tau\right) \quad ,$$

which are recognized as the retrogressive, constant attrition-rate, homogeneous-force model time solutions for the condition that Red group-1 wins. But these paths define C_4. Thus C_4 is a boundary of our region.

Paths terminating on the n_2-axis are parameterized by substituting $s_1 = 0$ into equations 65. We then obtain

$$m = \frac{\beta_2 s_2}{\sqrt{\alpha_1\beta_1}} \;\sinh\left(\sqrt{\alpha_1\beta_1}\;\tau\right)$$

$$n_1 = \frac{\beta_2 s_2}{\beta_1}\left[\cosh\left(\sqrt{\alpha_1\beta_1}\;\tau\right) - 1\right] \qquad (67)$$

$$n_2 = s_2 .$$

These paths satisfy the nonparametric equation

$$\beta_1 n_1^{\,2} + 2\beta_2 n_1 n_2 - \alpha_1 m^2 = 0 \quad , \qquad (68)$$

which defines a quadratic surface, denoted S, which intersects C_5 in the n_2-axis. Substituting $n_2 = 0$ into (68) yields

$$\beta_1 n_1^{\,2} - \alpha_1 m^2 = 0 \quad , \qquad (69)$$

which is the boundary between C_3 and C_4. Thus, paths terminating on C_5 lie in a region R_1 bounded by C_4, C_5 and S. Optimal trajectories in this region are illustrated in Figure 2.

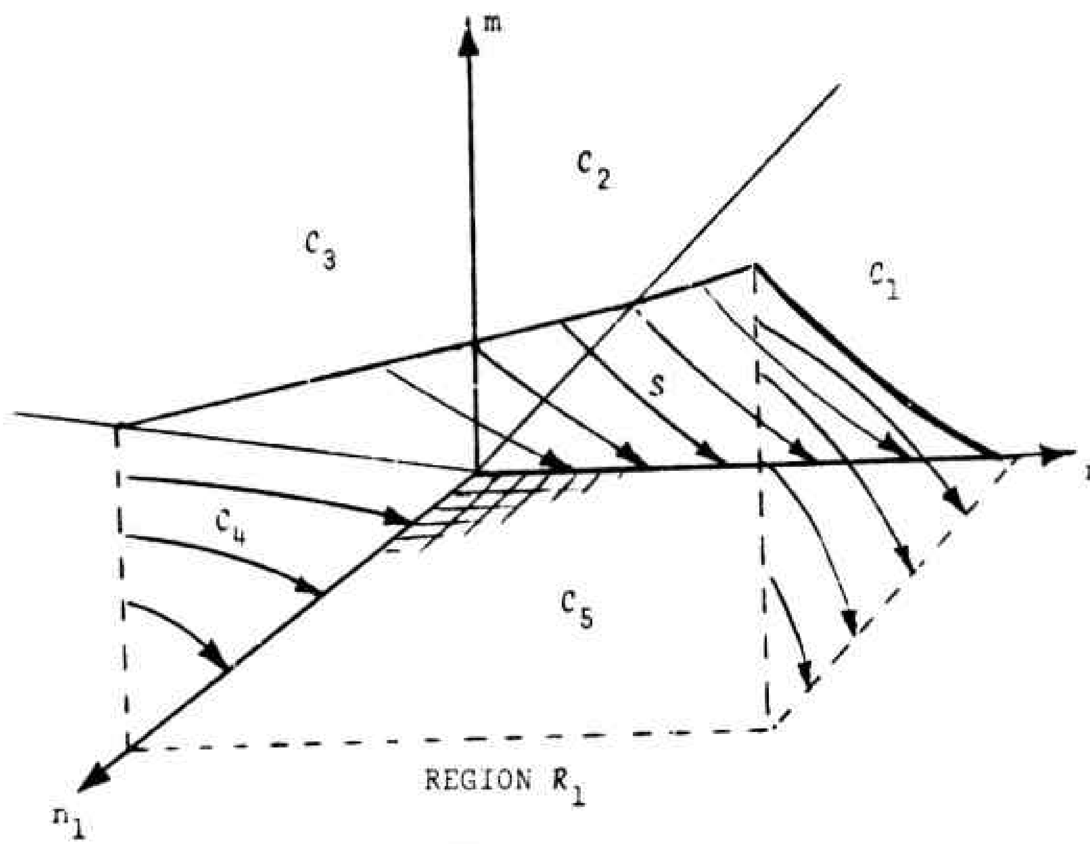


Figure 2

The remainder of our 3-space is bounded by C_1, C_2, C_3 and S. Optimal paths in this region must eventually terminate on C_1, C_2, or C_3. Consider any path in the interior of this region. Eventually, such a path must intersect a boundary of the region. Suppose an optimal interior trajectory intersects C_3 at time

τ_o at the point $m = r$, $n_1 = s_1$, $n_2 = 0$. We must conclude that at time $\tau_o + \Delta\tau$, $e^* = 0$, for if $e^* = 1$, the path would not reach C_3.

Now on C_3 we have

$$U = H = -\frac{c}{\sqrt{\alpha_1}} \sqrt{\alpha_1 r^2 - \beta_1 s_1^2}$$

so that

$$v(\tau_o) = -\frac{c\sqrt{\alpha_1}\, r}{\sqrt{\alpha_1 r^2 - \beta_1 s_1^2}} \tag{70}$$

and

$$w_1(\tau_o) = \frac{c\,\beta_1 s_1}{\sqrt{\alpha_1}\sqrt{\alpha_1 r^2 - \beta_1 s_1^2}}\,. \tag{71}$$

Since the main equation must be satisfied at the point $(r, s_1, 0)$ on C_3, we find w_2 to be

$$w_2(\tau_o) = \frac{c\sqrt{\alpha_1}\,\beta_1 s_1}{\alpha_2\sqrt{\alpha_1 r^2 - \beta_1 s_1^2}}\,. \tag{72}$$

From equations 71 and 72 it is seen that

$$\alpha_1 w_1(\tau_o) = \alpha_2 w_2(\tau_o)\,, \tag{73}$$

making $S(\tau_o) = 0$.

We may solve for the partials of the value function at $\tau_o + \Delta\tau$ from the retrogressive path equations with $e^* = 0$,

$$
\begin{aligned}
\dot{v} &= -\alpha_2 w_2 \\
\dot{w}_1 &= -\beta_1 v \qquad\qquad (74) \\
\dot{w}_2 &= -\beta_2 v,
\end{aligned}
$$

and obtain

$$
\begin{aligned}
v(\tau_o + \Delta\tau) &= v(\tau_o) - \alpha_2 w_2(\tau_o)\Delta\tau \\
w_1(\tau_o + \Delta\tau) &= w_1(\tau_o) - \beta_1 v(\tau_o)\Delta\tau \qquad\qquad (75) \\
w_2(\tau_o + \Delta\tau) &= w_2(\tau_o) - \beta_2 v(\tau_o)\Delta\tau.
\end{aligned}
$$

Then,

$$S(\tau_o + \Delta\tau) = \alpha_1 w_1(\tau_o + \Delta\tau) - \alpha_2 w_2(\tau_o + \Delta\tau)\ , \qquad (76)$$

which, using equation 73, reduces to

$$S(\tau_o + \Delta\tau) = -(\alpha_1\beta_1 - \alpha_2\beta_2)\ v(\tau_o)\Delta\tau. \qquad (77)$$

But $v(\tau_o)$ is negative on C_3, making $S(\tau_o + \Delta\tau)$ positive, which gives $e^* = 1$ by equation 61. But this contradicts our original assumption that $e^* = 0$. Hence, we must conclude that there are no optimal trajectories reaching C_3 except those that originate in C_3. Finally, it will be noted that our conclusion was reached independent of Cases I and II.

By a similar analysis we conclude that optimal paths cannot leave by way of S. We know that $e^* = 1$ yields paths paralleling S; thus, we may assume that if there existed an optimal trajectory intersecting S from above, $e^* = 0$. But $e^* = 1$ is optimal on S, hence, if $e^* = 0$ intersects S, then S must be a switching surface and $S = \alpha_1 w_1 - \alpha_2 w_2$ must therefore be zero on S. Again the analysis given in equations 74 through 77 applies yielding the contradiction that $S(\tau_o) = 0$ implies $e^* = 1$ at $\tau_o + \Delta\tau$.

We must conclude then that optimal paths can only leave R_2 by way of C_1 and C_2. The necessary form of such paths is obtained by integrating the retrogressive path equations with $e^* = 1$ and initial conditions $m = r$, $n_1 = 0$, $n_2 = s_2$. We then find

$$
\begin{aligned}
m &= r \cosh\left(\sqrt{\alpha_1\beta_1}\ \tau\right) + \frac{\beta_2 s_2}{\sqrt{\alpha_1\beta_1}} \sinh\left(\sqrt{\alpha_1\beta_1}\ \tau\right) \\
n_1 &= \frac{\beta_2 s_2}{\beta_1}\left[\cosh\left(\sqrt{\alpha_1\beta_1}\ \tau\right) - 1\right] \\
&\quad + \frac{\sqrt{\alpha_1\beta_1}\ r}{\beta_1} \sinh\left(\sqrt{\alpha_1\beta_1}\ \tau\right) \\
n_2 &= s_2
\end{aligned}
\tag{78}
$$

for all paths in the region bounded by C_1, C_2, C_3 and S.

The surface of paths terminating on the boundary between C_1 and C_2 may be found by setting $\sqrt{\alpha_2}\ r = \sqrt{\beta_2}\ s_2$ in (78) and eliminating the variable τ or by writing the state equation from the retrogressive path equations of equation 64 directly. The result is the cone K given by

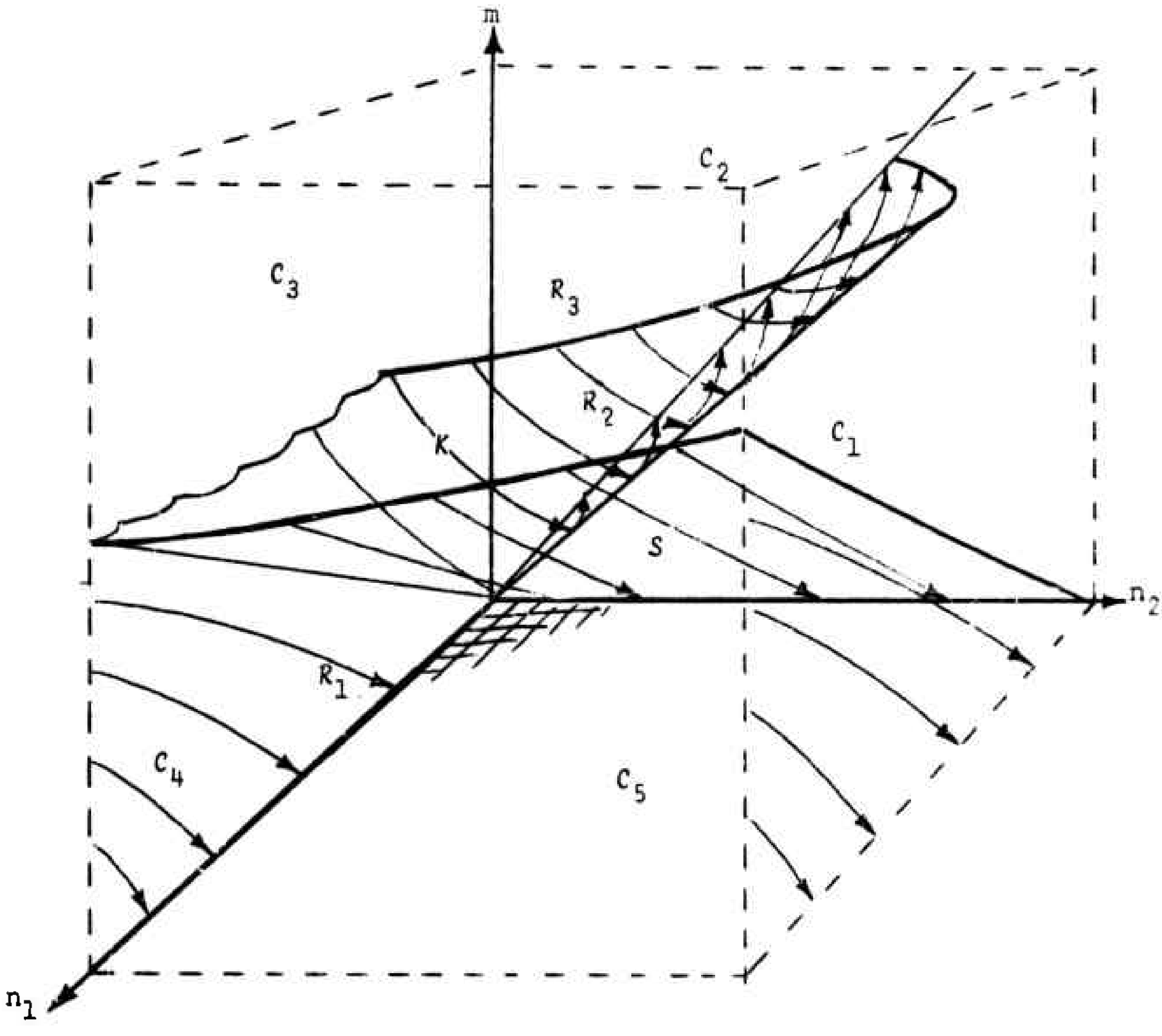
$$\alpha_1\alpha_2 m^2 - \alpha_2\beta_1 n_1^2 - 2\alpha_2\beta_2 n_1 n_2 - \alpha_1\beta_2 n_2^2 = 0 \ , \qquad (79)$$

which also intersects C in the boundary between C_3 and C_4, as can be seen by substituting $n_2 = 0$ into equation 79. For initial points above this cone, Blue wins, that is, Blue annihilates both Red weapon groups. Cone K partitions the region above S into R_2, which lies between S and K, and R_3, which lies above K .

The Case I solution is illustrated in Figure 3.

Case II

We now turn our attention to Case II and begin by considering paths terminating on C_5. In the neighborhood of C_5, $e^* = 0$ is optimal and the retrogressive path equations are therefore



CASE I BOUNDARY SURFACES

Figure 3

$$\dot{m} = \beta_1 n_1 + \beta_2 n_2 \qquad \dot{v} = -\alpha_2 w_2$$

$$\dot{n}_1 = 0 \qquad \dot{w}_1 = -\beta_1 v \qquad (80)$$

$$\dot{n}_2 = \alpha_2 m \qquad \dot{w}_2 = -\beta_2 v.$$

Solving equations 80 with C_5 parameterized as $m = 0$, $n_1 = s_1$, $n_2 = s_2$ and initial conditions 0, d_1 and d_2 for v, w_1 and w_2, respectively, yields

$$m = \frac{\beta_1 s_1 + \beta_2 s_2}{\sqrt{\alpha_2 \beta_2}} \sinh\left(\sqrt{\alpha_2\beta_2}\ \tau\right)$$

$$n_1 = s_1$$

$$n_2 = \frac{\beta_1 s_1 + \beta_2 s_2}{\beta_2} \cosh\left(\sqrt{\alpha_2\beta_2}\ \tau\right) - \frac{\beta_1 s_1}{\beta_2}$$

$$v = -\frac{d_2\sqrt{\alpha_2\beta_2}}{\beta_2} \sinh\left(\sqrt{\alpha_2\beta_2}\ \tau\right) \qquad (81)$$

$$w_1 = \frac{d_2\beta_1}{\beta_2}\left[\cosh\left(\sqrt{\alpha_2\beta_2}\ \tau\right) - 1\right] - d_1$$

$$w_2 = d_2 \cosh\left(\sqrt{\alpha_2\beta_2}\ \tau\right) .$$

We desire to establish the boundaries of the space swept out by optimal paths terminating on C_5. Substituting $s_1 = 0$ into equation 81 gives paths terminating on the n_2 - axis. These paths lie entirely in the m,n_2 plane since $s_1 = 0$ requires that $n_1 = 0$. These paths are described by

$$m = \frac{\beta_2 s_2}{\sqrt{\alpha_2\beta_2}} \sinh\left(\sqrt{\alpha_2\beta_2}\ \tau\right)$$

$$n_1 = 0 \tag{82}$$

$$n_2 = s_2 \cosh\left(\sqrt{\alpha_2\beta_2}\ \tau\right) .$$

Path terminating on the n_1-axis are parameterized by substituting $s_2 = 0$ into equation 81. The resulting equations are

$$m = \frac{\beta_1 s_1}{\sqrt{\alpha_2\beta_2}} \sinh\left(\sqrt{\alpha_2\beta_2}\ \tau\right),$$

$$n_1 = s_1 \tag{83}$$

$$n_2 = \frac{\beta_1 s_1}{\beta_2}\left[\cosh\left(\sqrt{\alpha_2\beta_2}\ \tau\right)^{-1}\right]$$

and they define a quadratic surface $\mathcal{U}$ whose nonparametric representation is

$$\beta_2 n_2^2 + 2\beta_1 n_1 n_2 - \alpha_2 m^2 = 0. \tag{84}$$

Finally, we encounter a transition surface T owing to the fact that S is negative on C_5 but $dS/d\tau$ is positive, leading eventually to $S = 0$. From equations 80 we obtain

$$\frac{dw_1}{dw_2} = \frac{\beta_1}{\beta_2}, \tag{85}$$

whose solution is

$$\beta_2(w_1 - d_1) = \beta_1(w_2 - d_2). \tag{86}$$

Equation 86 coupled with the fact that $\alpha_1 w_1 = \alpha_2 w_2$ on T results in

$$w_2 = \frac{\alpha_1(\beta_1 d_2 - \beta_2 d_1)}{(\alpha_1\beta_1 - \alpha_2\beta_2)} \tag{87}$$

on the transition surface. Then from equations 81 and 87,

$$\cosh\left(\sqrt{\alpha_2\beta_2}\;\tau\right) = \frac{\alpha_1(\beta_1 d_2 - \beta_2 d_1)}{d_2(\alpha_1\beta_1 - \alpha_2\beta_2)} \tag{88}$$

defines T in terms of τ. The existance of the switching surface is guaranteed because $\alpha_2 d_2 > \alpha_1 d_1$ and $\alpha_1\beta_1 > d_2\beta_2$ implies $\beta_1 d_1 > \beta_2 d_2$.

This makes the right-hand side of (88) greater than one. For values of τ greater than that τ satisfying equation 88, $S > 0$, making $e^* = 1$ the optimal tactic.

We will not compute the equation of the transition surface, but rather determine its boundries. To determine the intersection of T with the m,n_2 plane, we substitute equation 88 into equation 82. Let

$$\theta = \cosh\left(\sqrt{\alpha_2\beta_2}\ \tau\right) = \frac{\alpha_1(\beta_1 d_2 - \beta_2 d_1)}{d_2(\alpha_1\beta_1 - \alpha_2\beta_2)} \ . \tag{89}$$

Then the intersection is given by

$$\begin{aligned} m &= \frac{\beta_2 s_2}{\sqrt{\alpha_2\beta_2}} \sqrt{\theta^2 - 1} \\ n_2 &= s_2\theta \ , \end{aligned} \tag{90}$$

which is a parameterization of the straight line L given by

$$\frac{m}{n_2} = \sqrt{1 - \frac{1}{\theta^2}}\ \frac{\sqrt{\beta_2}}{\sqrt{\alpha_2}} \tag{91}$$

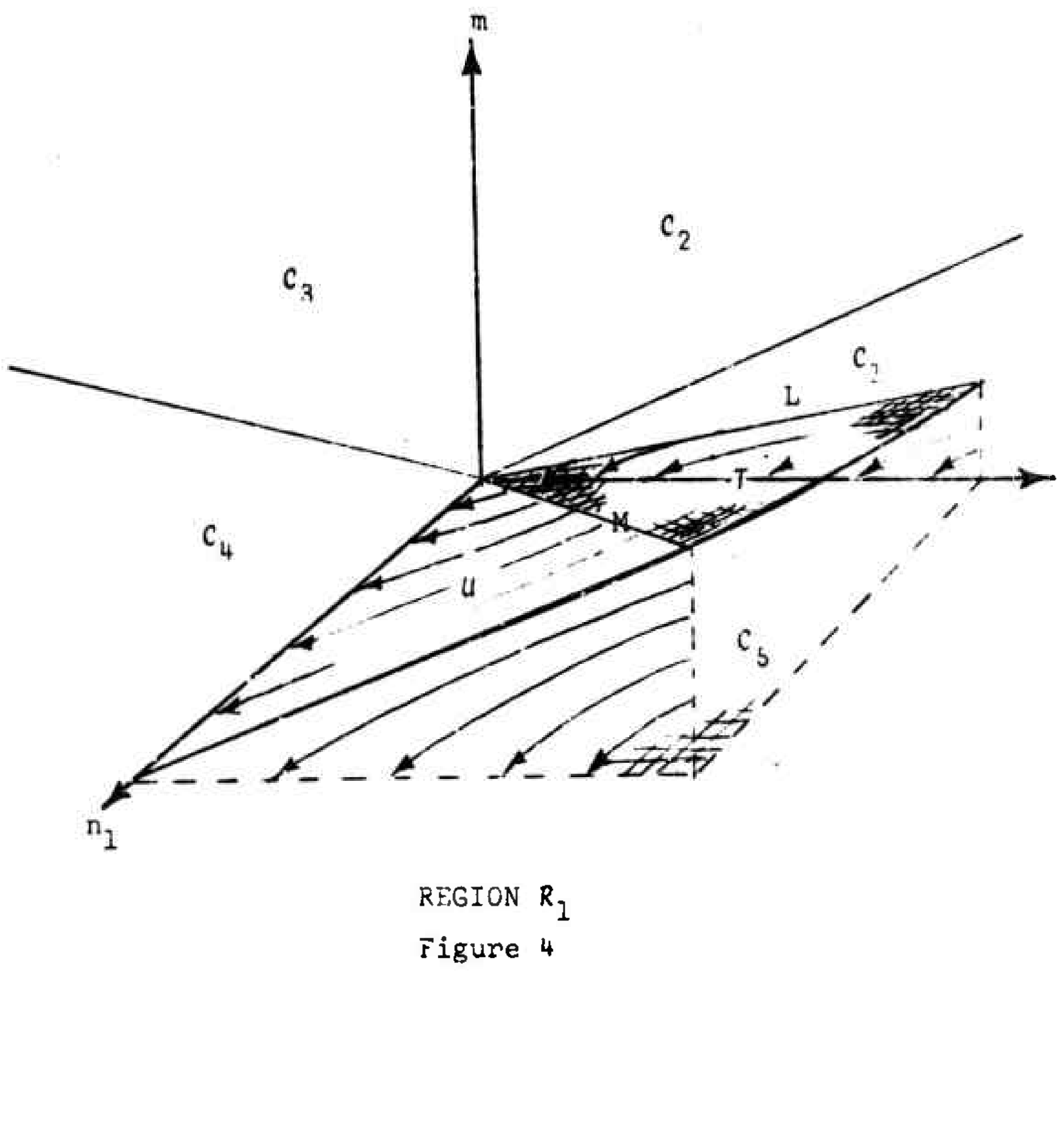
Noting that $\sqrt{1 - \frac{1}{\theta^2}}$ is less than one, the slope of L (which is dm/dn_2) is less than the slope of the boundary between C_1 and C_2. L therefore lies entirely in C_1.

In a similar fashion the intersection of T and U are described by equations 83 and 89 as

$$m = \frac{\beta_1 \sqrt{\theta^2 - 1}}{\sqrt{\alpha_2 \beta_2}} n_1$$

$$n_2 = \frac{\beta_1}{\beta_2} (\theta - 1) n_1 , \tag{92}$$

which is a ray M from the origin. The region R_1 bounded by C_5, T, and portions of C_1 and U is illustrated in Figure 4.



REGION R_1

Figure 4

Above the transition surface T, $e^* = 1$ is the optimal tactic. Optimal paths intersecting T fill a region R_2 whose boundries we now determine.

Optimal paths intersecting T in the line L satisfy the path equation 64 with initial conditions determined by equation 91 of the straight line L together with $y_1 = 0$. From equation 64 we obtain

$$\frac{dm}{dn_1} = \frac{\beta_1 n_1 + \beta_2 n_2}{\alpha_1 m} , \tag{93}$$

which upon integration with initial conditions taken on L becomes the surface S given by

$$\alpha_2\beta_1 n_1^{\ 2} + 2\alpha_2\beta_2 n_1 n_2 + \alpha_1\beta_2\left(1 - \frac{1}{\theta^2}\right)n_2^{\ 2} - \alpha_1\alpha_2 m^2 = 0. \tag{94}$$

Substituting $n_2 = 0$ into (94) shows that S intersects C in the boundary between C_3 and C_4 as well as in line L. S therefore forms a cone, above which the solutions of Cases I and II are identical.

Integrating equation 93 with initial conditions taken on the ray M completes the boundries of R_2. The resulting surface, denoted W, satisfies the equation

$$\beta_1\left[\frac{\alpha_1\beta_1}{\alpha_2\beta_2}(\theta^2 - 1) - 2(\theta - 1)\right]n_1^{\ 2} + 2\beta_2 n_1 n_2 - \alpha_1 m^2 = 0. \tag{95}$$

This surface also intersects the plane m, n_1 in the boundary between C_3 and C_4. We reason this since W contains optimal paths with $e^* = 1$ and intersects the origin. In the plane m, n_1 there is only one path which intersects the origin, and that is the path along the C_3, C_4 boundary. Therefore, the C_3, C_4 boundary lies in W. The boundries of region R_2 are therefore T, S, and W.

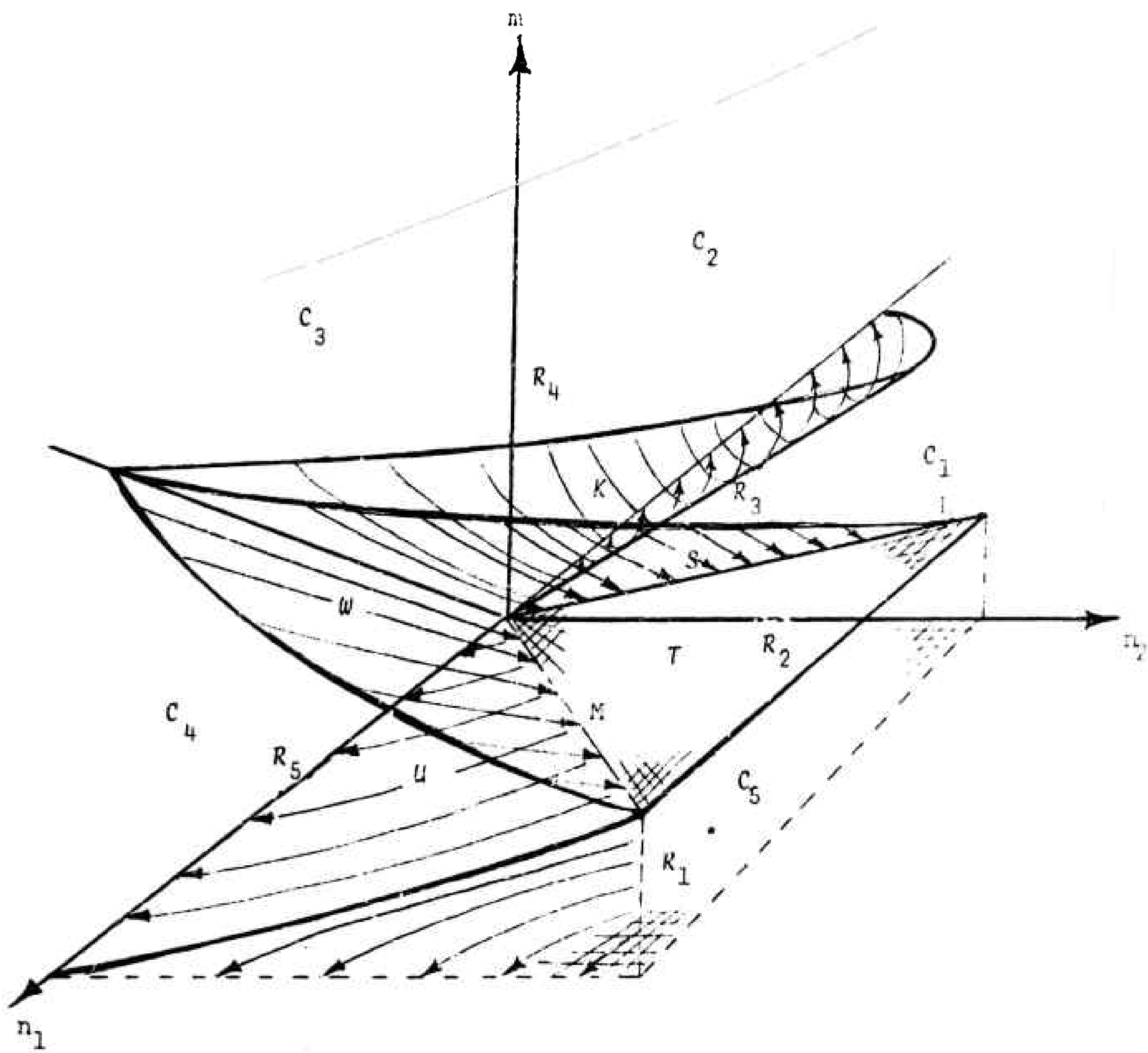
Above the surface S are regions R_3 and R_4, separated by the cone K described by equation 79. Figure 4 illustrates the bounding surfaces for Case II.

All paths are now accounted for except paths in the region R_5, bounded by U, W, and C_4. We again employ the logic that optimal paths in R_5 must eventually leave R_5. Our analysis proving that optimal paths cannot leave R_5 by way of C_4 is identical to our argument in Case I that optimal paths could not leave by way of C_3. On C_4 we have

$$U = H = \frac{d_1}{\sqrt{\beta_1}} \sqrt{\beta_1 s_1^2 - \alpha_1 r^2}$$

so that assuming optimal paths intersect C_4 at time τ_o with $e^* = 0$,

$$v(\tau_o) = -\frac{d_1 \alpha_1 r}{\sqrt{\beta_1}\sqrt{\beta_1 s_1^2 - \alpha_1 r^2}} \tag{96}$$



CASE II BOUNDARY SURFACES

Figure 5

and

$$w_1(\tau_o) = \frac{d_1 \beta_1 s_1}{\sqrt{\beta_1}\ \sqrt{\beta_1 s_1^{\,2} - \alpha_1 r^2}} \quad . \tag{97}$$

From the main equation we obtain

$$w_2(\tau_o) = \frac{d_1 \alpha_1 \beta_1 s_1}{\alpha_2 \sqrt{\beta_1}\ \sqrt{\beta_1 s_1^{\,2} - \alpha_1 r^2}} \quad , \tag{98}$$

which demonstrates that

$$\alpha_1 w_1(\tau_o) = \alpha_2 w_2(\tau_o) \tag{99}$$

on C_4.

The remainder of our argument is given by equations 74 through 77, which do not depend upon either Cases I or II, and hence we conclude that there are no optimal paths leaving R_5 by way of C_4.

There are no paths leaving R_5 by way of W, for $e^* = 1$ gives paths paralleling W and $e^* = 0$ gives paths moving away from W. We therefore must include that all paths leave R_5 by way of U with $e^* = 1$, for $e^* = 0$ gives paths paralleling U. The surface U is therefore a transition surface, at which point $e^* = 0$ becomes the optimal strategy. In the language of Isaacs (1965), U is a universal surface, meaning that paths intersecting U stay in U and finally intersect the n_1-axis. This is all formalized by demonstrating that indeed $S = 0$ on U.

Paths in $\mathcal{U}$ with $e^* = 0$ satisfy the main equation. Using the solutions of path equations 83 in $\mathcal{U}$ yields

$$v \frac{dm}{d\tau} + w_1 \frac{dn_1}{d\tau} + w_2 \frac{dn_2}{d\tau} = v\,\beta_1 s_1 \cosh\left(\sqrt{\alpha_2\beta_2}\,\tau\right) + \frac{w_2\beta_1 s_1\sqrt{\alpha_2\beta_2}}{\beta_2} \sinh\left(\sqrt{\alpha_2\beta_2}\,\tau\right) = 0. \tag{100}$$

Paths in C_5 with $e^* = 1$ also satisfy the main equation. Consider such a path intersecting $\mathcal{U}$ at the point P at time τ_o where P satisfies equation 83. The main equation $\tau_o + \Delta\tau$ becomes

$$v\beta_1 s_1 \cosh\left(\sqrt{\alpha_2\beta_2}\,\tau\right) + w_1\alpha_1 m = v\beta_1 s_1 \cosh\left(\sqrt{\alpha_2\beta_2}\,\tau\right) + \frac{w_1\alpha_1\beta_1 s_1}{\sqrt{\alpha_2\beta_2}} \sinh\left(\sqrt{\alpha_2\beta_2}\,\tau\right) = 0. \tag{101}$$

Subtracting equation 101 from equation 100 gives

$$\frac{w_2\beta_1 s_1\sqrt{\alpha_2\beta_2}}{\beta_2} \sinh\left(\sqrt{\alpha_2\beta_2}\,\tau\right) - \frac{w_1\alpha_1\beta_1 s_1}{\sqrt{\alpha_2\beta_2}} \cdot \sinh\left(\sqrt{\alpha_2\beta_2}\,\tau\right) = 0. \tag{102}$$

which reduces to

$$\alpha_2 w_2 - \alpha_1 w_1 = 0 , \tag{103}$$

which proves our contention that U is a switching surface.

2.9 *References*

Isaacs, R., *Differential Games*, New York: John Wiley & Sons, Inc., 1965.

Chapter 3

NUMERICAL SOLUTION PROCEDURE, VARIABLE-COEFFICIENT MODEL

George Cooper and George Miller

This chapter presents a simplified numerical solution procedure for solving the general heterogeneous-force battle model described by equations 1 and 2 in Chapter 2, Part A. The procedure is essentially a recursive time-stop solution of the attrition equations. The model describes spacially distributed forces, by groups, with the Blue force defending and the Red force assaulting. The forces are assumed to be of approximately battalion size. They may employ both direct- and indirect-fire systems such as small arms, personnel carriers, tanks, antitank guns and guided missiles, mortars, artillery, and rockets. The general flow of the model operation is given in Figure 1. Specific operating details, inputs, rules of engagement, and the computer program are given in the following sections.

3.1 Attrition

The attrition equations are approximated in the computer computations by[1]

$$\Delta N_J \approx -\sum_I A_{IJ} E_{IJ} I_{IJ} M_I \Delta T \qquad J = 1,2,\ldots,JJ \qquad (1)$$

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[1]Notation changes have been made in this section to facilitate consistency with the computer program.

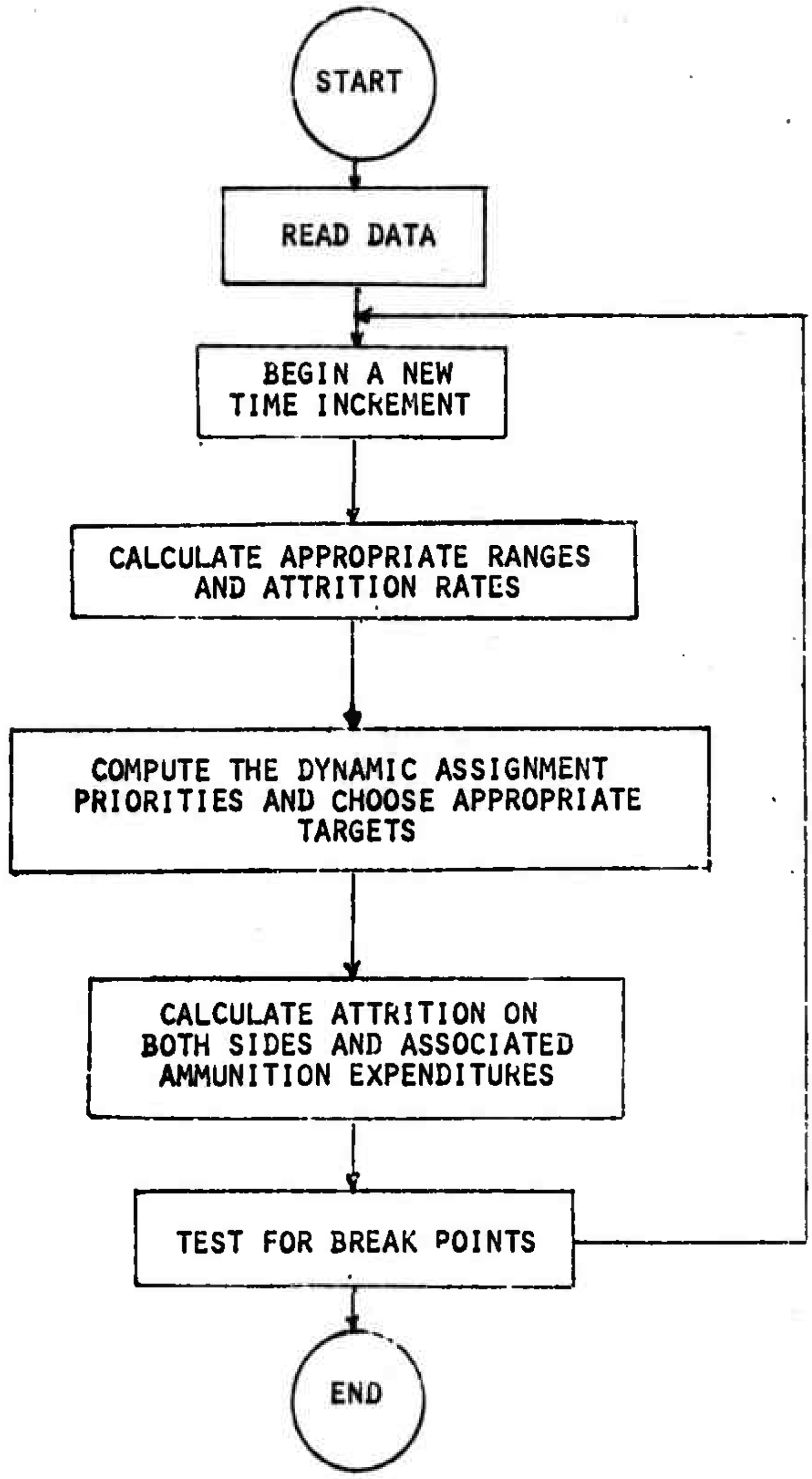

Figure 1 Overall Flow Diagram for Numerical Solution Procedure

$$\Delta M_I \simeq - \sum_J B_{JI}H_{JI}K_{JI}N_JF_J\Delta T \quad I = 1,2,\ldots,II \ , \quad (2)$$

where

$A_{IJ}(B_{JI})$ = the attrition rate for the I^{th} (J^{th}) Blue (Red) weapon or the J^{th} (I^{th}) Red (Blue) weapon,

ΔM_I = the number of Blue I-group losses in the time increment ΔT,

ΔN_J = the number of Red J-group losses in the time increment ΔT,

$E_{IJ}(H_{JI})$ = Blue (Red) allocation factors,

$I_{IJ}(K_{JI})$ = Blue (Red) intelligence factors,

F_J = the average fraction of time that the J-type weapons are not advancing.

The F_J factor is included to account for the fact that advancing weapons usually do not fire. The model assumes that attrition is a continuous process, and accordingly, the attrition coefficients are adjusted to account for the down firing time.

The attrition of forces throughout the battle is determined by recursively computing the losses over the discrete time increment ΔT using the surviving numbers of units M_I and N_J at the the end of the previous time increment. The reader is reminded that, for area-lethality systems such as artillery, the attrition rate is itself a function of the surviving number of targets [B, 6.0].

It was noted in Section 1.2, Part B, that the attrition rates are functions of range. This feature is incorporated

by assuming that the attrition rates (computed by the methods described in Part B) are polynomial functions of the range, R_{IJ}, between force groups I and J. A polynomial regression of the form

$$A_{IJ}(R_{IJ}) = A0_{IJ} + A1_{IJ}R_{IJ} + A2_{IJ}R_{IJ}^2 + A3_{IJ}R_{IJ}^3 + A4_{IJ}R_{IJ}^4 \quad (3)$$

is used. The coefficients of this 4^{th}-order polynomial are input data for the model.

Following the results obtained in the previous chapter, the assignment parameters E_{IJ} and H_{JI} are (0,1) in the model. They are, however, implemented as single arrays E_I and H_J. E_I is the index number of the target on which Blue group I is assigned to fire, and H_J is the index of the target on which Red group J is assigned to fire.

3.2 Range Considerations

The initial X and Y coordinates of each weapon group are input data to the model. These are used to calculate the ranges between weapon groups. The Blue defenders are assumed to be stationary; hence, their coordinates do not change. The Red attackers are assumed to attack due north; hence, only their Y coordinates change. Further, all Red weapon groups which move are assumed to move at the same constant velocity.

The basic program computes slant ranges R_{IJ} between I and J groups at each time increment. A switch is provided in the program, however, which facilitates the placing of all Blue units essentially on line on the FEBA. It implies that each group in a force fires on its assigned target group, which is located at a range DFEBA(J) from it rather than the slant-range R_{IJ}. DFEBA(J) is the distance of Red group J from FEBA. This parameter is included in the periodic engagement status reports of the basic program as an indicator of the approximate positions of the Red forces. If all the Red forces begin the engagement at positions equidistant from the FEBA, then use of the switch has the effect of each weapon group firing on its assigned target group, which is located directly in front of it. This procedure simplifies the model and reduces the computation time.

3.3 *Area-Fire Effects*

Weapon groups located in the same vicinity are given the same (X,Y) coordinates for purposes of range computations. Hence, for area-fire weapons, the attrition is calculated not only for the target specifically assigned but for any other targets which have the same location. Under the assumption that weapon groups do not shield each other from area-fire weapon effects, the appropriate attrition rates are independently applied to each target group.

3.4 Ammunition Considerations

Ammunition expenditures are calculated at each time increment. Required input data are the firing rates for each (I,J) combination, as 4^{th}-order polynomial functions of range.[1] For all but indirect firing weapons, the firing rates are easily obtained from the attrition rates for impact-lethality systems. For indirect-fire, area-lethality systems, the firing rates are inputs to the attrition-rate model, and assumed known.

3.5 Rules of Engagement

3.5.1 Blue Target Assignments

At the beginning of each time increment, all Blue groups are assigned a target. The new target may be the same as the previous one, a different target or the "null" target, i.e., no target at all. Eligible targets are comprised of those targets within range which are not externally prohibited. Targets within range are those within the open-fire range and beyond the cease-fire range. Externally prohibited targets for a given weapon group are those for which a zero attrition rate has been submitted.

An assignment is made among the eligible targets according to one of two procedures. The first procedure is that the weapon-target combination in question have the highest product $A_{IJ}B_{JI}$ for all eligible Red targets of group J. This criterion is an approximation to the marginal effectiveness measures developed by S. Sternberg in the previous chapter. If all

[1]See Section 3.1.

eligible targets are unable to return fire, then all of the above products will be zero. In this case, Blue group I is assigned to fire on that J for which A_{IJ} is maximum. An optional assignment via a priority table P(I,J) is also available. This table is constructed as follows. For each Blue group I, one enters in the corresponding row of the P matrix the Red groups J in order of decreasing priority. Attrited weapon groups are not assigned a target, and live weapon groups are not assigned to fire at dead ones.

3.5.2 *Red Target Assignments*

After each Blue group is assigned a Red target for the duration of the time increment, each Red group is assigned to fire on either an eligible Blue target or on the null targets. To be eligible, a Blue target must be within range (by the definition of 3.5.1), not externally prohibited, and must be detectable. A Blue weapon group is not detectable by a Red weapon group unless it is firing, or unless the Red group has advanced to within a specified range, RSTAR, of the Blue group. First consideration is given to selecting a target among those which are both eligible and firing on the Red group in question. Among such targets, Red group J is assigned to fire on the Blue group I for which the product $B_{JI}A_{IJ}$ is the highest. If Red group J is not being fired upon, then it fires at the eligible target I for which $B_{JI}A_{IJ}$

is the highest. If all eligible Blue groups are unable to return fire, then all the above products will be zero. In this case, the eligible target for which B_{JI} is the highest is selected. If there are no eligible targets, the null target is assigned. An optional assignment via a priority table Q(J,I) is also available. This table is constructed as follows . For each Red group J, one enters in the corresponding row of the Q matrix the Blue groups I in order of decreasing priority. Attrited weapon groups are not assigned a target, and live weapon groups are not assigned to fire on dead ones.

3.6 *Stopping Rules*

A number of optional stopping rules are provided for in the model. These include

(1) The Red Attackers Cross the FEBA

The distance of each Red force group from the FEBA, initially given as input, is decremented and checked after each time step. When any one of the Red force groups reaches the FEBA, the engagement is terminated. The model does not include final protective fires or the commitment of reserves.

(2) Red Attrition Limit Reached

A force group may be designated as a break group or nonbreak group in the input data. At the end of each time step, the total percentage of casualties for all break groups is checked against the specified allowable percentage. If the allowable percentage of cas-

ualties is exceeded, the attacking force breaks and the engagement is terminated.

(3) Blue Attrition Limit Reached

The Blue defender's attrition is measured in the same manner as for the attackers. When the allowable percentage of critical or "break group" forces is exceeded, the Blue defense breaks and the engagement is terminated.

(4) Engagement Time Limit Exceeded

When a specified maximum duration for the engagement is exceeded, it is terminated regardless of current force strengths or positions.

3.7 Input Data[1]

Data cards must be submitted in the order shown in this section. The varaibles must be in the order shown on each card, and in the FORTRAN format indicated in parenthesis after each variable name. An attempt has been made to indicate the number of cards of each type required using the notation [ ] to denote truncated integer division. Meters and seconds have been taken as the standard units for distance and time. As long as the user is consistent throughout a data set, however, any other units may be used. Only the printed comments depend on the units chosen. All cards begin with column one.

---

[1]A glossary of symbols is given in Section 3.9.

(1) Header Card

II(I3),JJ(I3),DELTAT(F6.1),V(F7.3),
RSTAR(F6.1),NSTEP(13),BBREAK(F3.3),
RBREAK(F3.3),TSTOP(F7.2),LINE(I1),
PRIOF(I1),ENGRAN(F6.1),RUNIT(I3)

There is only one header card required.

(2) Blue Force Specifications

M(F4.0),XI(F7.1),YI(F7.1),IM(I1),
BBT(I1),(1X),BID(A8)

There should be "II" of these cards.

(3) Red Force Specifications

N(F4.0),XJ(F7.1),YJ(F7.1),IN(I1),RBT(I1),
(1X),RID(A8),4X,STAT(I1)

There should be "JJ" of these cards

(4) Distance from FEBA

DFEBA(F6.1)

There should be 12 of these per card for "[JJ/12]+1" cards.

(5) Attrition Rate Modifier to Reflect Movement

ARMTRM(F4.3)

There should be 18 of these per card for "[JJ/18]+1" cards.

(6) Blue Open-Fire Ranges

S(F6.1)

For each successive Blue group, there should be a set of "[JJ/12]+1" cards containing the open-fire ranges (12 per card) against each Red group.

(7) Red Open-Fire Ranges

T(F6.1)

For each successive Red group, there should be a set of "[II/12]+1" cards containing the open-fire ranges (12 per card) against each Blue group.

(8) Blue Cease-Fire Ranges

BCFIRE(F6.1)

There should be 12 of these per card for "[II/12]+1" cards.

(9) Red Cease-Fire Ranges

RCFIRE(F6.1)

There should be 12 of these per card for "[JJ/12]+1" cards.

(10) Blue Attrition and Firing Rates

A0(E8.3),A1(E8.3),A2(E8.3),A3(E8.3),A4(E8.3),
AA0(E8.3),AA1(E8.3),AA2(E8.3),AA3(E8.3),AA4(E8.3)

There should be "II x JJ" of these cards. They are read in sets of "JJ" for each successive Blue group.

(11) Red Attrition and Firing Rates

B0(E8.3),B1(E8.3),B2(E8.3),B3(E8.3),B4(E8.3),
BB0(E3.3),BB1(E8.3),BB2(E8.3),BB3(E8.3),
BB4(E8.3)

There should be "JJ x II" of these cards. They are read in sets of "II" for each successive Red group.

(12) Blue Intelligence Coefficient

BI(F4.3)

For each successive Blue group there should appear 18 per card for "[JJ/18]+1" cards.

(13) Red Intelligence Coefficient

RI(F4.3)

For each successive Red group there should appear 18 per card for "[II/18]+1" cards.

(14) Blue Priority Table (Required if PRIOR≠0)

P(II,JJ)

For each Blue group there should be at least one card [Format (36I2,8X)] containing up to 36 Red group numbers arranged in decreasing order according to their priority as targets for the Blue group.

(15) Red Priority Table (Required if PRIOR≠0)

Q(JJ,II)

For each Red group there should be at least one card [Format (36I2,8X)] containing 36 Blue group numbers arranged in decreasing order according to their priority as targets for the Red group.

3.8 Interpretation of Output Engagement Status Reports

At user specified intervals, engagement status reports are printed. Such reports are also given before the engagement begins, and at the break point. Proper interpretation of these reports depends on some knowledge of the program logic. Key points to remember are the following:

(1) No computations are carried out before the initial report. Hence, this should reflect the input data

if the data was correctly submitted.

(2) Since no targets are assigned to attrited weapon groups, a target assigned to an obviously annihilated weapon groups implies that that group was annihilated in the current time increment.

(3) Since attrited weapon groups do not advance, the distance from the FEBA for such groups reflects the approximate distance at which that group becomes annihilated.

(4) Distance from the FEBA is a guideline only, and does not necessarily reflect the range between a Red group and any Blue group. A unit may be very close to the FEBA, but still out of range of some of the blue weapon groups. An exception to this rule is when LINE is nonzero.

(5) Because of area-fire effects, some forces may be noticed to decrease in numbers, even though they have not directly been the target of any weapon.

(6) The number of time increments between status reports is specified as input and is multiples of time increments in which actual battle activity takes place. Increments in which no losses occur are not counted for purposes of determining when the next printout occurs.

3.9 *Program Glossary*

3.9.1 *Dimension Variables*

An "I" subscript always implies that I = 1,...,II. A "J" subscript always implies that J = 1,...,JJ. Both II and JJ must not exceed 40. Hence, all the arrays are restricted accordingly

ARRAY NAME	DEFINITION
A(I,J)	The Blue on Red attrition rate
A0(I,J),A1(I,J),A2(I,J), A3(I,J),A4(I,J)	The 0^{th} through 4^{th} coefficients, respectively, of a 4^{th}-order polynomial to predict A(I,J) as a function of range
AA0(I,J),AA1(I,J), AA2(I,J),AA3(I,J), AA4(I,J)	The 0^{th} through 4^{th} coefficients, respectively, of a 4^{th}-order polynomial to predict Blue firing rates as a function of range
ARMTRM(J)	A fraction to reflect the proportion of time that Red group J fires while advancing
B(J,I)	The Red on Blue attrition rate

B0(J,I),B1(J,I), B2(J,I),B3(J,I), B4(J,I)	The 0^{th} through 4^{th} coefficients, respectively, of a 4^{th}-order polynomial to predict Red attrition rates as a function of range
BB0(J,I),BB1(J,I), BB2(J,I),BB3(J,I), BB4(J,I)	The 0^{th} through 4^{th} coefficient, respectively, of a 4^{th}-order polynomial to predict Red firing rates as a function of range
BBT(I)	A flag telling whether or not Blue group I is a "break group," 1 = yes, 0 = no
BCFIRE(I)	The minimum range at which Blue group I can fire
BEXP(I)	The cumulative Blue ammunition expenditures
BI(I)	The Blue intelligence coefficients
BID(I)	Contains an 8-byte alphabetic code for identifying Blue group I
DFEBA(J)	Initially, the data giving the distance of Red force J from the FEBA, updated each time step

E(I)	The Red target number of Blue group I
H(J)	The Blue target number of Red group J
IM(I)	Mode of fire of Blue group I, 1 = direct, 0 = indirect
IN(J)	Mode of fire of Red group J, 1 = direct, 0 = indirect
K(I,J)	Range Constant = $[XI(I) - XJ(J)]^2$
M(I)	Number of surviving Blue group I
MM(I)	Number of surviving Blue group I at the end of the previous time step
N(J)	Number of surviving Red group J
R(I,J)	Range between Blue I and Red J
RBT(J)	A flag telling whether or not Red group J is a "break group," 1 = yes, 0 = no
RCFIRE(J)	The minimum range at which Red group J is allowed to fire
REXP(J)	Cumulative Red ammunition expenditures

P(I,J)	Blue on Red Priority Table, P(I,J) = Red target group corresponding to I and J, where I denotes the Blue weapon group and J denotes the priority of the Red target group to which Blue is to be assigned (J=1, denotes the highest priority)
Q(J,I)	Red on Blue Priority Table, Q(J,I) = Blue target group corresponding to I and J, where J denotes the Red target group and I denotes the priority of the blue target to which the Red group is to be assigned
RI(J)	The Red intelligence coefficients
RID(J)	Contains an 8-byte alphabetic code for identifying Red group J
S(I,J)	Range[1] at which Blue I can open fire on Red J
STAT(J)	A flag telling whether or not a Red group J remains stationary during the engagement, 1 = yes, 0 = no
T(J,I)	Range[1] at which Red J can open fire on Blue I

---

[1]These variables are used to reflect (a) open-fire ranges based on doctrine and/or (b) open-fire ranges based on weapon capabilities, and should not exceed the ranges considered in the attrition rates.

XI(I)	X position of Blue group I
XJ(J)	X position of Red group J
YI(I)	Y position of Blue group I
YJ(J)	Y position of Red group J

3.9.2 Nondimension Variables

VARIABLE	DEFINITION
BBREAK	The fraction of "break groups" which must be lost for Blue to break off the engagement
DELTAT	The size of the time step in seconds
ENGRAN	Range at which the engagement begins
II	The total number of Blue weapon groups
JJ	The total number of Red weapon groups
LINE	If nonzero, causes DFEBA(J) to be used for the range between Red group J and all Blue groups I
NSTEP	The number of time increments between printings of the engagement status report
RBREAK	The fraction of "break groups" which must be lost for Red to break off the engagement

RSTAR	The range within which a Red group may detect a Blue target, whether or not the target is firing
RUNIT	The index of the *mobile* Red weapon group which opens the engagement when its distance to the FEBA is less than the engagement range (ENGRAN).
PRIOR	If nonzero, the priority tables given by P(40,40) and Q(40,40) are used to make weapon assignments
TSTOP	The input time (in seconds) at which the engagement will terminate if no other break points are reached
V	The average velocity of the attack

3.10 Program Logic

Numbers in parentheses refer to program line numbers.

(1,81) Read data

(82,100) Initialize

- (82,87) Add up the initial numbers of "break" groups
- (88,93) Zero out ammunition expenditures
- (94,97) Print out initial forces
- (98,100) Calculate range constants

(101,154) Prepare for assignments

- (101,105) Update clock, check for time limit exceeded

(106,109) Zero out all assignments

(110,115) Update DFEBA and Y coordinates

(117,136) If the line switch is not on, calculate slant ranges and attrition rates between all pairs of eligible, nondefunct targets

(137,154) If the line switch is on, calculate range by equating it with DFEBA, and calculate attrition rates between all pairs of eligible, nondefunct targets

(155,193) Make blue on Red assignments

(155,166) Assign using Priority Table P(II,JJ)

(167,178) Test for greatest "Sternberg coefficient"

(179,187) Test for greatest attrition rate (if necessary)

(188,193) Compute Blue ammunition expenditures

(194,242) Make Red on Blue assignments

(194,206) Assign using Priority Table Q(JJ,II)

(207,222) Test for greatest "Sternberg coefficient" among those firing at you

(223,228) Test for greatest "Sternberg coefficient" among those detectable targets

(229,242) If "Sternberg coefficient" is all zeros, test for greatest attrition rate

(243,249) Calculate Red ammunition expenditures

(250,261) Combat activity check

If no combat this time increment, check for crossing of the FEBA. If FEBA is crossed, terminate the engagement, otherwise return to beginning (90,142).

If combat takes place, proceed to calculate attrition for this time step.

(262,279) Calculate Red's attrition of Blue

(262,268) Direct fire

(269,272) Area fire (target intended plus any nearby)

(273,279) Set to zero any Blue force that was annihilated

(280,297) Calculate Blue's attrition of Red

(280,286) Direct fire

(287,290) Area fire (target intended plus any nearby)

(291,297) Set to zero any Red force that was annihilated

(298,309) Check for break points and engagement status reports

Test for crossing of the FEBA

Test for Red attrition limit reached

Test for Blue attrition limit reached

Test to see if it is time to print an engagement status report

(310,320) Set up flags for the printing of break point comments

(321,354) Print out engagement status report with breakpoint comment if any, and either halt the program or return for another time increment, as necessary

8.11 Program Listing

Lines 1-13 were used for program identification in The University of Michigan computer system.

```
14          INTEGER B(40),R(40),BBT(40),RBT(40),STAT(40),PRIOR
15          INTEGER P(40,40),Q(40,40),RUNIT
16          REAL*8 BID(40),RID(40)
17          DIMENSION S(40,40),T(40,40)
18          DIMENSION DFEBA(40),XI(40),XJ(40),YI(40),YJ(40)
19          DIMENSION IM(40),IN(40)
20          DIMENSION BCFIRE(40),RCFIRE(40),BEXP(40),REXP(40)
21          DIMENSION R1(40,40),B1(40,40)
22          DIMENSION A0(40,40),A1(40,40),A2(40,40),A3(40,40),A4(40,40)
23          DIMENSION B0(40,40),B1(40,40),B2(40,40),B3(40,40),B4(40,40)
24          DIMENSION AA0(40,40),AA1(40,40),AA2(40,40),AA3(40,40)
25          DIMENSION AA4(40,40),BB0(40,40),BB1(40,40),BB2(40,40)
26          DIMENSION BB3(40,40),BB4(40,40)
27          DIMENSION A(40,40),B(40,40)
28          DIMENSION R(40,40)
29          DIMENSION ARMTRM(40)
30          REAL MM(40),M(40),N(40),K(40,40)
31          READ(5,5)II,JJ,DELTAT,V,RSTAR,NSTEP,BBREAK,RBREAK,TSTOP,LINE,PRIOR
32         1,ENGRAN,RUNIT
33    5     FORMAT(2I3,F6.1,F7.3,F6.1,I3,2F3.3,F7.2,2I1,F6.1,I3)
34          BPCT=BBREAK*100.
35          RPCT=RBREAK*100.
36          WRITE(6,6)II,JJ,DELTAT,V,RSTAR,NSTEP,BPCT,RPCT,TSTOP,LINE
37         1,PRIOR,ENGRAN,RUNIT
38    6     FORMAT('1','GROUND COMBAT BETWEEN ',I3,
39         1  ' BLUE (DEFENDERS) WEAPON TYPES',/,'AND',I3,
40         2  ' RED (ATTACKERS) WEAPON TYPES.',/,'THE TIME STEP IS ',
41         3  F6.1,' SECONDS.',/,'THE VELOCITY (AVERAGE) IS',
42         4  F7.3,' METERS PER SECOND',/,'RSTAR IS',F6.1,' METERS',/,
43         5  'THERE WILL BE',I3,' STEPS BETWEEN PRINTOUTS.',/,
44         6  'BLUE BREAK POINT IS',F5.1,' PERCENT',/,'RED BREAK ',
45         7 'POINT IS',F5.1,' PERCENT',/,'THE ENGAGEMENT WILL NOT',
46         8 ' EXCEED',F10.2,' SECONDS',/,'LINE= ',I5,/,
47         9'PRIOR= ',I3,/,'ENGRAN= ',F6.1,' METERS',/,'RUNIT= ',I3,/,/,/)
48          DO 10 KK=1,II
49    10    READ(5,15) M(KK),XI(KK),YI(KK),IM(KK),BBT(KK),BID(KK)
50    15    FORMAT(F4.0,2F7.1,2I1,1X,A8,5X,I1)
51          DO 20 KK=1,JJ
52    20    READ(5,15) N(KK),XJ(KK),YJ(KK),IN(KK),RBT(KK),RID(KK),STAT(KK)
53          READ(5,32) (CHECK(J),J=1,JJ)
```

```
          READ(5,41) (ARMTRM(J),J=1,JJ)
          DO 26 I=1,II
   26     READ(5,32) (S(I,J),J=1,JJ)
          DO 28 J=1,JJ
   28     READ(5,32) (T(J,I),I=1,II)
          READ(5,32) (BCFIRE(I),I=1,II)
          READ(5,32) (RCFIRE(J),J=1,JJ)
          DO 36 I=1,II
          DO 36 J=1,JJ
   36     READ(5,40) A0(I,J),A1(I,J),A2(I,J),A3(I,J),A4(I,J)
         1  ,AA0(I,J),AA1(I,J),AA2(I,J),AA3(I,J),AA4(I,J)
          DO 37 J=1,JJ
          DO 37 I=1,II
   37     READ(5,40) B0(J,I),B1(J,I),B2(J,I),B3(J,I),B4(J,I)
         1  ,BB0(J,I),BB1(J,I),BB2(J,I),BB3(J,I),BB4(J,I)
   40     FORMAT(10F6.3)
          DO 33 I=1,II
   33     READ(5,41) (BI(I,J),J=1,JJ)
          DO 34 J=1,JJ
   34     READ(5,41) (RI(J,I),I=1,II)
   41     FORMAT(18F4.3,8X)
   32     FORMAT(12F6.1,8X)
          IF (PRIOR .EQ. 0) GO TO 49
          DO 46 I=1,II
   46     READ(5,48) (P(I,J),J=1,JJ)
          DO 47 J=1,JJ
   47     READ(5,48) (Q(J,I),I=1,II)
   48     FORMAT(36I2,8X)
   49     BTEST = 0.
          DO 50 I=1,II
   50     BTEST=BTEST+BBT(I)*M(I)
          RTEST=0.
          DO 60 J=1,JJ
   60     RTEST=RTEST+RBT(J)*N(J)
          DO 200 I=1,II
          E(I)=0
  200     BEXP(I)=0.0
          DO 210 J=1,JJ
          H(J)=0
  210     REXP(J)=0.0
          TI    = 0.
          ISTOP =4
          GO TO 1300
   70     CONTINUE
          DO 100 K1=1,II
          DO 100 K2=1,JJ
  100     K(K1,K2) = (XI(K1) - XJ(K2))**2
  500     TI    = TI    + DELTAT
          IF(TI.LE.TSTOP) GO TO 505
          TI=TI-DELTAT
          ISTOP=5
          GO TO 1300
  505     DO 510 L=1,II
  510     E(L) = 0
          DO 520 L =1,JJ
  520     H(L)=0
          DO 525 J=1,JJ
          IF(N(J).EQ.0.) GO TO 525
          IF(STAT(J).EQ.1) GO TO 525
          DFEBA(J)=DFEBA(J)-V*DELTAT
```

```
      YJ(J)=YJ(J)+V*DELTAT
  525 CONTINUE
      IF (DFEBA(KUNIT) .GT. ENGRAN) GO TO 500
      IF(LINE.NE.0) GO TO 2000
      DO 530 I=1,II
      DO 530 J=1,JJ
      IF(M(I).EQ.0..OR.N(J).EQ.0.) GO TO 529
      R(I,J)=SQRT(R(I,J)+(YJ(J)-Y(I))*(YJ(J)-Y(I)))
      IF(R(I,J).LE.BCF(RE(I)).OR.R(I,J).GE.S(I,J)) GO TO 526
      A(I,J)=A0(I,J)+A1(I,J)*R(I,J)+A2(I,J)*R(I,J)**2+
     1 A3(I,J)*R(I,J)**3+A4(I,J)*R(I,J)**4
      GO TO 527
  526 A(I,J)=0.
  527 IF(R(I,J).LE.RCF(RE(J)).OR.R(I,J).GE.T(J,I)) GO TO 528
      B(J,I)=B0(J,I)+B1(J,I)*R(I,J)+B2(J,I)*R(I,J)**2+
     1 B3(J,I)*R(I,J)**3+B4(J,I)*R(I,J)**4
      GO TO 530
  528 B(J,I)=0.
      GO TO 530
  529 A(I,J)=0.
      B(J,I)=0.
  530 CONTINUE
      GO TO 430
 2000 DO 2500 I=1,II
      DO 2500 J=1,JJ
      IF(M(I).EQ.0..OR.N(J).EQ.0.) GO TO 2300
      R(I,J)=DFEBA(J)
      IF(R(I,J).LE.BCF(RE(I)).OR.R(I,J).GE.S(I,J)) GO TO 2200
      A(I,J)=A0(I,J)+A1(I,J)*R(I,J)+A2(I,J)*R(I,J)**2+
     1 A3(I,J)*R(I,J)**3+A4(I,J)*R(I,J)**4
      GO TO 2250
 2200 A(I,J)=0.
 2250 IF(R(I,J).LE.RCF(RE(J)).OR.R(I,J).GE.T(J,I)) GO TO 2275
      B(J,I)=B0(J,I)+B1(J,I)*R(I,J)+B2(J,I)*R(I,J)**2+
     1 B3(J,I)*R(I,J)**3+B4(J,I)*R(I,J)**4
      GO TO 2500
 2275 B(J,I)=0.
      GO TO 2500
 2300 A(I,J)=0.
      B(J,I)=0.
 2500 CONTINUE
  430 IF (PRIOR .EQ. 0) GO TO 534
      DO 450 I=1,II
      E(I) = 0
      IF (M(I) .EQ. 0.) GO TO 450
      DO 440 J=1,JJ
      IF (P(I,J) .EQ. 0) GO TO 440
      IF (A(I,P(I,J)) .LE. 0.) GO TO 440
      E(I) = P(I,J)
      GO TO 450
  440 CONTINUE
  450 CONTINUE
      GO TO 555
  534 DO 550 I=1,II
      TEM1=0.
      TEM2=0.
      I1=0
      I2=0
      DO 540 J=1,JJ
      IF(A(I,J).EQ.0.) GO TO 540
```

```
               TEM=A(I,J)*B(J,I)
               IF(TEM.LE.TEM1) GO TO 540
               TEM1=TEM
               I1=J
               GO TO 540
       535     IF(A(I,J).LE.TEM2) GO TC 540
               TEM2=A(I,J)
               I2=J
       540     CONTINUE
               IF(I1.EQ.0) GO TC 545
               E(I)=I1
               GC TO 550
       545     E(I)=I2
       550     CONTINUE
      555      DC 59C I=1,II
               IF(E(I)) 590,590,589
      589      RATE=AA0(I,E(I))+AA1(I,E(I))*R(I,E(I))+AA2(I,E(I))*R(I,E(I))
              1 **2+AA3(I,E(I))*R(I,E(I))**3+AA4(I,E(I))*R(I,E(I))**4
               BEXP(I)=BEXP(I)+RATE*M(I)*DELTAT
      590      CCNTINUE
               IF (PRIOR .EQ. 0) GO TO 595
               DO 594 J=1,JJ
               H(J) = 0
               IF (N(J) .EQ. 0.) GO TO 594
               DC 593 I=1,II
               IF (E(I) .EQ. 0 .AND. R(I,J) .GT. RSTAR) GO TO 593
               IF (Q(J,I) .EQ. 0) GO TC 593
               IF (B(J,Q(J,I)) .LE. C.) GC TO 593
               H(J) = Q(J,I)
               GC TO 594
      593      CCNTINUE
      594      CCNTINUE
               GC TC 701
      595      DO 700 J=1,JJ
               I1=0
               I2=0
               I3=0
               RTEM1=0.
               RTEM2=0.
               RTEM3=0.
               DC 675 I=1,II
               IF(B(J,I).EQ.0.) GO TC 675
               IF(A(I,J).EQ.0.) GO TC 650
               TEM=B(J,I)*A(I,J)
               IF(E(I).NE.J) GC TO 6CC
               IF(TEM.LE.RTEM1) GO TC 675
               RTEM1=TEM
               I1=I
               GC TO 675
      60C      IF(E(I).GT.0) GO TO 610
               IF(R(I,J).GT.RSTAR) GC TO 675
      61C      IF(TEM.LE.RTEM2) GO TC 675
               RTEM2=TEM
               I2=I
               GC TO 675
      650      IF(E(I).GT.0) GO TO 66C
               IF(R(I,J).GT.RSTAR) GC TO 675
      660      IF(B(J,I).LE.RTEM3) GC TC 675
               RTEM3=B(J,I)
               I3=I
```

```
  675 CONTINUE
      IF(I1.EQ.0) GO TO 680
      H(J)=I1
      GO TO 700
  680 IF(I2.EQ.0) GO TO 690
      H(J)=I2
      GO TO 700
  690 H(J)=I3
  700 CONTINUE
 701  DO 705 J=1,JJ
      IF(H(J)) 705,705,702
 702  RATE=BB0(J,H(J))+BB1(J,H(J))*R(H(J),J)+BB2(J,H(J))*R(H(J),J)
     1 **2+BB3(J,H(J))*R(H(J),J)**3+BB4(J,H(J))*R(H(J),J)**4
      RATE=RATE*ARMTRM(J)
      REXP(J)=REXP(J)+RATE*N(J)*DELTAT
 705  CONTINUE
      ISUM = 0
      DO 725 I=1,II
  725 ISUM=ISUM+E(I)
      IF(ISUM) 730,730,1000
  730 DO 750 J=1,JJ
  750 ISUM = ISUM+H(J)
      IF(ISUM) 800,800,1000
 800  DO 850 J=1,JJ
      KK=J
      IF(DFEBA(J).LE.0.0) GO TO 1230
 850  CONTINUE
      GO TO 500
 1000 DO 1100 I=1,II
      MM(I)=M(I)
      DO 1050 J=1,JJ
      IF (IN(J).EQ.0) GO TO 1030
      IF (H(J).NE.I) GO TO 1050
      M(I)=M(I)-B(J,I)*N(J)*DELTAT*RI(J,I)*ARMTRM(J)
      GO TO 1050
 1030 IF(H(J).EQ.I) GO TO 1040
      IF(XI(H(J)).NE.XI(I).OR.YI(H(J)).NE.YI(I)) GO TO 1050
 1040 M(I)=M(I)-B(J,I)*N(J)*MM(I)*DELTAT*RI(J,I)*ARMTRM(J)
 1050 CONTINUE
      IF(M(I).GT.0.) GO TO 1100
      M(I)=0.
      IF(PRIOR.EQ.0) GO TO 1100
      DO 1070 J=1,JJ
      DO 1070 LL=1,II
 1070 IF(Q(J,LL).EQ.I) Q(J,LL)=0
 1100 CONTINUE
      DO 1200 J=1,JJ
      NN=N(J)
      DO 1150 I=1,II
      IF(IM(I).EQ.0) GO TO 1130
      IF (E(I).NE.J) GO TO 1150
      N(J)=N(J)-A(I,J)*MM(I)*DELTAT*BI(I,J)
      GO TO 1150
 1130 IF(E(I).EQ.J) GO TO 1140
      IF(XJ(E(I)).NE.XJ(J).OR.YJ(E(I)).NE.YJ(J)) GO TO 1150
 1140 N(J)=N(J)-A(I,J)*MM(I)*NN*DELTAT*BI(I,J)
 1150 CONTINUE
      IF(N(J).GT.0.)GO TO 1200
      N(J)=0.
      IF(PRIOR.EQ.0) GO TO 1200
```

```
      DO 1170 I=1,II
      DO 1170 LL=1,JJ
 1170 IF (P(I,LL) .EQ. J) P(I,LL)=0
 1200 CONTINUE
      TEST=0.
      DO 1210 J=1,JJ
      KK=J
      IF(DFEBA(J).LE.0.) GO TO 1230
 1210 TEST=TEST+N(J)*RBT(J)
      IF(1.-TEST/RTEST.GE.RBREAK) GO TO 1240
      TEST=0.
      DO 1220 I=1,II
 1220 TEST=TEST+M(I)*BBT(I)
      IF(1.-TEST/BTEST.GE.BBREAK) GO TO 1250
      NTEST=NTEST+1
      IF(NTEST.NE.NSTEP) GO TO 500
      ISTOP=0
      GO TO 1300
 1230 ISTOP=1
      DO 1235 MMM=KK,JJ
      IF(DFEBA(MMM).GT.0.0) GO TO 1235
      DFEBA(MMM)=0.0
 1235 CONTINUE
      GO TO 1300
 1240 ISTOP=2
      GO TO 1300
 1250 ISTOP=3
 1300 WRITE(6,1310) TI
 1310 FORMAT('1SURVIVING FORCES AT TIME = ',F5.0,' SECONDS:'
     + //,1X,'  BLUE TYPE',4X,'NUMBER',4X,'ROUNDS EXPENDED',3X,
     1 'TARGET',/)
      WRITE(6,1330) (I,BID(I),M(I),BEXP(I),E(I),I=1,II)
      WRITE(6,1320)
 1320 FORMAT('0',' RED TYPE',5X,'NUMBER',4X,'ROUNDS EXPENDED',
     + 3X,'TARGET',3X,'DISTANCE FROM FEBA',/)
      WRITE(6,1331) (J,RID(J),N(J),REXP(J),H(J),DFEBA(J),J=1,JJ)
 1330 FORMAT(' ',I2,1X,A8,3X,F6.2,7X,F8.2,10X,I2)
 1331 FORMAT(' ',I2,1X,A8,3X,F6.2,7X,F8.2,10X,I2,9X,F8.2)
      NTEST=0
      IF(ISTOP.EQ.0) GO TO 500
      IF(ISTOP.NE.1) GO TO 1340
      WRITE(6,1335)
 1500 CONTINUE
      WRITE(6,1550)
 1550 FORMAT('1')
 1335 FORMAT('0***BREAK POINT: RED FORCES HAVE REACHED FEBA',/)
      STOP
 1340 IF(ISTOP.NE.2) GO TO 1350
      WRITE(6,1345)
 1345 FORMAT('0***BREAK POINT: RED ATTRITION LIMIT REACHED'/'1')
      STOP
 1350 IF(ISTOP.NE.3) GO TO 1360
      WRITE(6,1355)
 1355 FORMAT('0***BREAK POINT: BLUE ATTRITION LIMIT REACHED'/'1')
      STOP
 1360 IF(ISTOP.NE.5) GO TO 70
      WRITE(6,1375)
 1375 FORMAT('0***BREAK POINT: ENGAGEMENT TIME LIMIT EXCEEDED')
      WRITE(6,1550)
      STOP
```

506

354 END
END OF FILE

3.12 *Program Enrichment*

1.0) Eliminate the 4^{th}-order polynomials to predict attrition rates as a function of range and those used to predict firing rates as a function of range. In using the program to study engagements over a variety of ranges, one must be very careful not to exceed the limiting ranges used in the fitting of the 4^{th}-order polynomials for the attrition and firing rates. An alternative to the above approach might be the use of linear, or higher order, interpolation in tables of attrition rates and firing rate versus range; including error returns whenever the appropriate ranges are exceeded.

2.0) Print out the current value of the attrition coefficient in the interim engagement data currently being printed.

3.0) Introduce variable velocities for the different Red weapons groups.

4.0) Allow both the X and Y coordinates of the Red attackers to change during an engagement.

5.0) Include, perhaps statistically, the effects of the intervening terrain.

3.13 *Sample Application*

The numerical solution procedure was applied to a hypothetical tactical situation. The application employed hypothetical numbers for the weapon systems and is only intended to indicate the kinds of results that can be obtained with the solution procedure.

Figure 2 portrays a Blue defensive tactical situation. Figures 3-13 are computer printouts for the application. Figure 3 indicates the total number of weapon groups on each side and values for a number of the model parameters. Figure 4 gives the number of survivors, by group, at time zero, and thus indicates the initial numbers of forces in the engagement. Figures 5-12 provide results at intermediate points in the battle. Figure 13 presents the results at the end of the battle, which in this case was due to the Red forces having reached the FEBA.

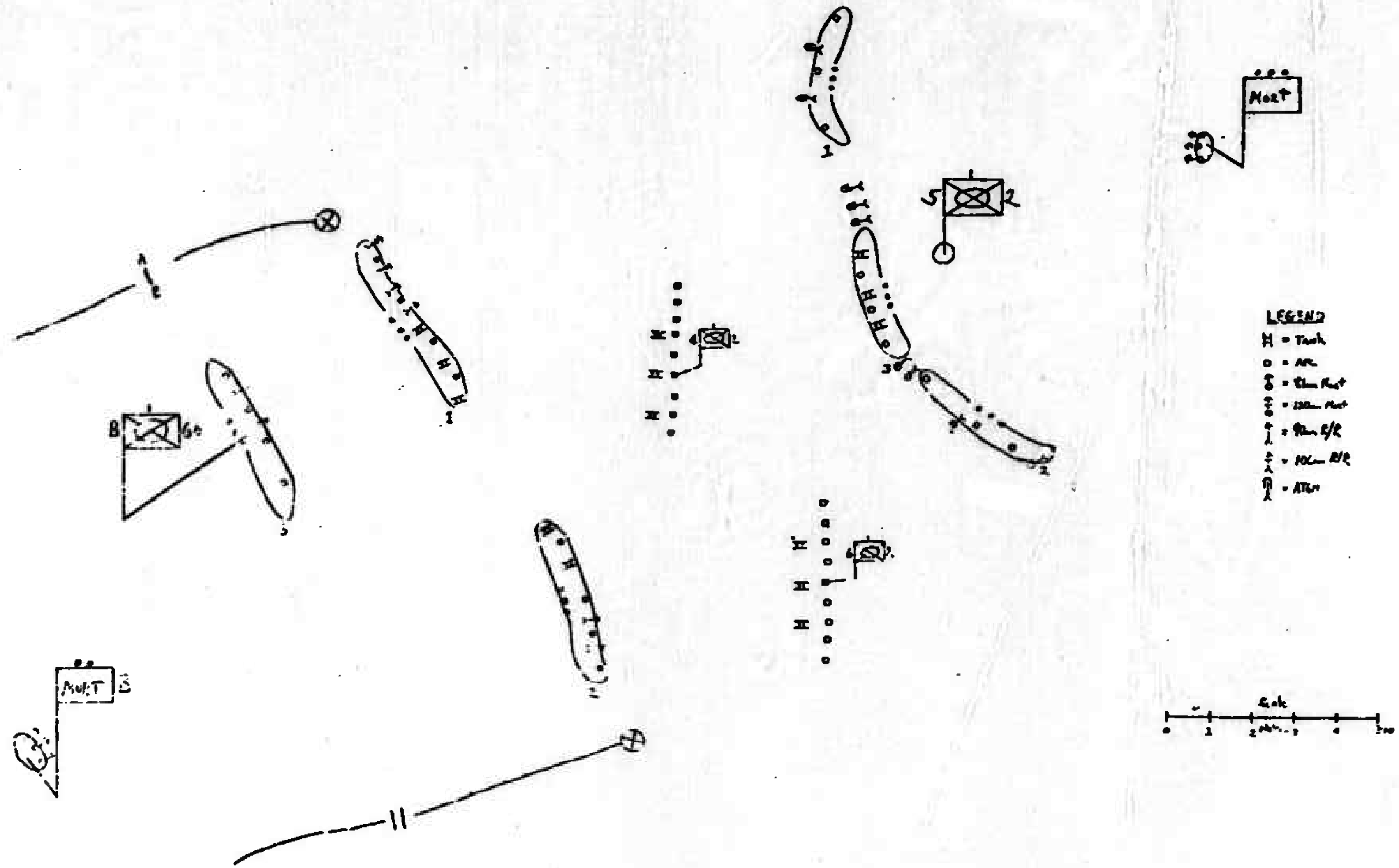

Figure 2 Idealized Target Model, Red Mounted Attack

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GROUND COMBAT BETWEEN  11 BLUE (DEFENDERS) SEARCH GROUPS
AND 12 RED (ATTACKERS) SEARCH GROUPS.
THE TIME STEP IS   10.0 SECONDS.
THE VELOCITY (AVERAGE) IS  2.000 METERS PER SECOND
RSTAR IS 700.0 METERS
THERE WILL BE  6 STEPS BETWEEN PRINTOUTS.
BLUE BREAK POINT IS 70.0 PERCENT
RED BREAK POINT IS 55.0 PERCENT
THE ENGAGEMENT WILL NOT EXCEED   2500.00 SECONDS
LINE=   0
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Figure 3

SURVIVING FORCES AT TIME = 0. SECONDS:

	BLUE GROUP	NUMBER	ROUNDS EXPENDED	TARGET
1	1B:TANK	3.00	0.0	0
2	1B:APC	4.00	0.0	0
3	1B:RR 90	2.00	0.0	0
4	1B:RRIC6	2.00	0.0	0
5	2B:TANK	2.00	0.0	0
6	2B:APC	4.00	0.0	0
7	2B:RR 90	2.00	0.0	0
8	3B:APC	4.00	0.0	0
9	3B:RR 90	2.00	0.0	0
10	MORT 81	3.00	0.0	0
11	AFAAC 37	3.00	0.0	0

	RED GROUP	NUMBER	ROUNDS EXPENDED	TARGET	DISTANCE FROM FEBA
1	4:TANK	3.00	0.0	0	1000.00
2	4:APC	5.00	0.0	0	1000.00
3	6:TANK	3.00	0.0	0	1000.00
4	6:APC	9.00	0.0	0	1000.00
5	1/5:APC	3.00	0.0	0	1050.00
6	1/5:ATGM	2.00	0.0	0	1050.00
7	2/5:APC	3.00	0.0	0	1180.00
8	3/5:TANK	3.00	0.0	0	1000.00
9	3/5:APC	3.00	0.0	0	1000.00
10	3/5:ATGM	2.00	0.0	0	1000.00
11	ATGM	3.00	0.0	0	1000.00
12	ATGM	2.00	0.0	0	1080.00
13	MORT 120	3.00	0.0	0	2000.00

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Figure 4

SURVIVING FORCES AT TIME = 60. SECONDS:

BLUE GROUP	NUMBER	ROUNDS EXPENDED	TARGET
1 1B:TANK	2.55	4.64	1
2 1B:APC	3.82	169.49	2
3 1B:RR 90	2.00	0.0	0
4 1B:RR106	0.83	1.49	2
5 2B:TANK	1.88	3.55	3
6 2B:APC	3.82	304.36	4
7 2B:RR 90	2.00	0.0	0
8 3B:APC	4.00	0.0	0
9 3B:RR 90	2.00	0.0	0
10 MORT 81	3.00	5.00	13
11 AFAAC 37	3.00	0.0	0

RED GROUP	NUMBER	ROUNDS EXPENDED	TARGET	DISTANCE FROM FEBA
1 4:TANK	2.83	5.91	1	880.00
2 4:APC	8.77	56.40	4	880.00
3 6:TANK	2.64	5.64	5	880.00
4 6:APC	8.83	355.28	6	880.00
5 1/5:APC	3.00	34.35	4	1050.00
6 1/5:ATGM	0.0	0.77	0	1050.00
7 2/5:APC	3.00	314.95	6	1190.00
8 3/5:TANK	3.00	11.88	1	1000.00
9 3/5:APC	3.00	36.12	4	1000.00
10 3/5:ATGM	0.0	0.78	0	1000.00
11 ATGM	0.0	1.76	0	1000.00
12 ATGM	0.0	0.38	0	1080.00
13 MORT 120	3.00	5.54	10	2000.00

Figure 5

SURVIVING FORCES AT TIME = 120. SECONDS:

BLUE GROUP	NUMBER	ROUNDS EXPENDED	TARGET
1 1B:TANK	2.20	9.79	1
2 1B:APC	3.80	546.74	2
3 1B:RR 90	1.80	0.67	2
4 1B:RR106	0.0	1.88	0
5 2B:TANK	1.77	7.87	3
6 2B:APC	3.67	674.14	4
7 2B:RR 90	1.79	0.67	4
8 3B:APC	4.00	0.0	0
9 3B:RR 90	2.00	0.0	0
10 MORT 81	3.00	17.99	13
11 AFAAC 37	3.00	0.0	0

RED GROUP	NUMBER	ROUNDS EXPENDED	TARGET	DISTANCE FROM FEBA
1 4:TANK	2.36	11.12	1	760.00
2 4:APC	8.35	142.43	3	760.00
3 6:TANK	2.28	10.57	5	760.00
4 6:APC	8.46	647.69	7	760.00
5 1/5:APC	3.00	115.42	2	1050.00
6 1/5:ATGM	0.0	0.77	0	1050.00
7 2/5:APC	3.00	629.99	6	1140.00
8 2/5:TANK	3.00	23.76	1	1000.00
9 3/5:APC	3.00	116.29	2	1000.00
10 3/5:ATGM	0.0	0.78	0	1000.00
11 ATGM	0.0	1.76	0	1000.00
12 ATGM	0.0	0.38	0	1080.00
13 MORT 120	3.00	11.88	10	2000.00

Figure 6

SURVIVING FORCES AT TIME = 180. SECONDS:

BLUE GROUP	NUMBER	ROUNDS EXPENDED	TARGET
1 1B:TANK	1.85	14.43	1
2 1B:APC	3.69	660.53	2
3 1B:RR 90	1.16	2.30	2
4 1B:RR106	0.0	1.88	0
5 2B:TANK	1.66	11.76	3
6 2B:APC	3.61	978.66	4
7 2B:RR 90	1.15	2.29	4
8 3B:APC	4.00	167.83	2
9 3B:RR 90	2.00	0.0	0
10 MORT 81	3.00	26.99	13
11 AFAAC 37	3.00	6.46	2

RED GROUP	NUMBER	ROUNDS EXPENDED	TARGET	DISTANCE FROM FEBA
1 4:TANK	1.92	15.42	1	640.00
2 4:APC	7.65	255.37	3	640.00
3 6:TANK	1.91	14.78	5	640.00
4 6:APC	7.92	762.40	7	640.00
5 1/5:APC	3.00	430.04	2	1050.00
6 1/5:ATGM	0.0	0.77	0	1050.00
7 2/5:APC	3.00	944.93	6	1180.00
8 3/5:TANK	3.00	35.64	1	1000.00
9 3/5:APC	3.00	416.64	2	1000.00
10 3/5:ATGM	0.0	0.78	0	1000.00
11 ATGM	0.0	1.76	0	1000.00
12 ATGM	0.0	0.38	0	1080.00
13 MORT 120	3.00	17.82	10	2000.00

Figure 7

SURVIVING FORCES AT TIME = 240. SECONDS:

BLUE GROUP	NUMBER	ROUNDS EXPENDED	TARGET
1 1B:TANK	1.52	18.55	1
2 1B:APC	3.58	1111.60	2
3 1B:RR 90	0.50	3.28	2
4 1B:RR106	0.0	1.88	0
5 2B:TANK	1.56	15.65	3
6 2B:APC	3.56	1225.83	4
7 2B:RR 90	0.45	3.24	4
8 3B:APC	4.00	680.25	2
9 3B:RR 90	2.00	0.0	0
10 MORT 81	3.00	35.98	13
11 AFAAC 37	3.00	24.60	2

RED GROUP	NUMBER	ROUNDS EXPENDED	TARGET	DISTANCE FROM FEBA
1 4:TANK	1.53	18.85	1	520.00
2 4:APC	6.63	336.67	3	520.00
3 6:TANK	1.54	18.26	5	520.00
4 6:APC	7.34	848.68	7	520.00
5 1/5:APC	3.00	744.66	2	1050.00
6 1/5:ATGM	0.0	0.77	0	1050.00
7 2/5:APC	3.00	1259.90	6	1180.00
8 3/5:TANK	3.00	47.52	1	1000.00
9 3/5:APC	3.00	716.99	2	1000.00
10 3/5:ATGM	0.0	0.78	0	1000.00
11 ATGM	0.0	1.76	0	1000.00
12 ATGM	0.0	0.38	0	1080.00
13 MORT 120	3.00	23.75	10	2000.00

Figure 8

SURVIVING FORCES AT TIME = 300. SECONDS:

BLUE GROUP	NUMBER	ROUNDS EXPENDED	TARGET
1 1B:TANK	1.19	22.08	1
2 1B:APC	3.44	1310.04	2
3 1B:RR 90	0.0	3.55	0
4 1B:RR106	0.0	1.88	0
5 2B:TANK	1.48	19.57	3
6 2B:APC	3.42	1423.77	4
7 2B:RR 90	0.0	3.44	0
8 3B:APC	4.00	1139.65	2
9 3B:RR 90	2.00	0.0	0
10 MORT 81	3.00	44.56	3
11 AFAAC 37	3.00	40.90	2

RED GROUP	NUMBER	ROUNDS EXPENDED	TARGET	DISTANCE FROM FEBA
1 4:TANK	1.19	21.63	1	400.00
2 4:APC	5.48	408.86	2	400.00
3 6:TANK	1.17	21.00	5	400.00
4 6:APC	6.71	945.54	6	400.00
5 1/5:APC	3.00	1059.28	2	1050.00
6 1/5:ATGM	0.0	0.77	0	1050.00
7 2/5:APC	3.00	1574.88	6	1180.00
8 3/5:TANK	3.00	59.40	1	1000.00
9 3/5:APC	3.00	1017.35	2	1000.00
10 3/5:ATGM	0.0	0.78	0	1000.00
11 ATGM	0.0	1.76	0	1000.00
12 ATGM	0.0	0.38	0	1080.00
13 MORT 120	3.00	29.65	10	2000.00

Figure 9

SURVIVING FORCES AT TIME = 360. SECONDS:

BLUE GROUP	NUMBER	ROUNDS EXPENDED	TARGET
1 1B:TANK	0.89	24.99	1
2 1B:APC	3.13	1462.71	2
3 1B:RR 90	0.0	3.55	0
4 1B:RR106	0.0	1.88	0
5 2B:TANK	1.41	23.55	3
6 2B:APC	3.10	1575.61	4
7 2B:RR 90	0.0	3.44	0
8 3B:APC	4.00	1561.43	2
9 3B:RR 90	2.00	0.0	0
10 MORT 81	2.99	53.99	3
11 AFAAC 37	3.00	55.54	2

RED GROUP	NUMBER	ROUNDS EXPENDED	TARGET	DISTANCE FROM FEBA
1 4:TANK	0.91	23.75	1	280.00
2 4:APC	4.14	500.73	2	280.00
3 6:TANK	0.78	22.99	5	280.00
4 6:APC	6.00	1064.63	6	280.00
5 1/5:APC	3.00	1373.90	2	1050.00
6 1/5:ATGM	0.0	0.77	0	1050.00
7 2/5:APC	3.00	1889.85	6	1180.00
8 3/5:TANK	3.00	71.28	1	1000.00
9 3/5:APC	3.00	1317.70	2	1000.00
10 3/5:ATGM	0.0	0.78	0	1000.00
11 ATGM	0.0	1.76	0	1000.00
12 ATGM	0.0	0.38	0	1080.00
13 MORT 120	3.00	35.62	10	2000.00

Figure 10

SURVIVING FORCES AT TIME = 420. SECONDS:

BLUE GROUP	NUMBER	ROUNDS EXPENDED	TARGET
1 1B:TANK	0.57	27.20	1
2 1B:APC	2.84	1580.47	2
3 1B:RR 90	0.0	3.55	0
4 1B:RR106	0.0	1.88	0
5 2B:TANK	1.36	27.65	3
6 2B:APC	2.75	1691.59	4
7 2B:RR 90	0.0	3.44	0
8 3B:APC	4.00	1923.94	2
9 3B:RR 90	2.00	2.09	2
10 MORT 81	2.99	62.93	3
11 AFAAC 37	3.00	68.92	2

RED GROUP	NUMBER	ROUNDS EXPENDED	TARGET	DISTANCE FROM FEBA
1 4:TANK	0.69	25.35	1	160.00
2 4:APC	2.27	554.45	2	160.00
3 6:TANK	0.38	24.20	5	160.00
4 6:APC	5.22	1153.41	6	160.00
5 1/5:APC	3.00	1688.52	2	1050.00
6 1/5:ATGM	0.0	0.77	0	1050.00
7 2/5:APC	3.00	2204.83	6	1180.00
8 3/5:TANK	3.00	83.16	1	1000.00
9 3/5:APC	3.00	1618.05	2	1000.00
10 3/5:ATGM	0.0	0.78	0	1000.00
11 ATGM	0.0	1.76	0	1000.00
12 ATGM	0.0	0.38	0	1080.00
13 MORT 120	3.00	41.56	10	2000.00

Figure 11

SURVIVING FORCES AT TIME = 490. SECONDS:

BLUE GROUP	NUMBER	ROUNDS EXPENDED	TARGET
1 1B:TANK	0.28	28.63	1
2 1B:APC	2.63	1676.56	2
3 1B:RR 90	0.0	3.55	0
4 1B:RR106	0.0	1.88	0
5 2B:TANK	1.34	31.96	3
6 2B:APC	2.37	1792.42	4
7 2B:RR 90	0.0	3.44	0
8 3B:APC	4.00	2226.25	2
9 3B:RR 90	2.00	4.26	2
10 MORT 81	2.99	71.91	3
11 AFAAC 37	3.00	81.48	2

RED GROUP	NUMBER	ROUNDS EXPENDED	TARGET	DISTANCE FROM FEBA
1 4:TANK	0.59	26.59	1	40.00
2 4:APC	0.31	576.02	2	40.00
3 6:TANK	0.0	24.60	5	40.00
4 6:APC	4.39	1221.13	6	40.00
5 1/5:APC	3.00	2003.14	2	1050.00
6 1/5:ATGM	0.0	0.77	0	1050.00
7 2/5:APC	3.00	2519.80	6	1180.00
8 3/5:TANK	3.00	99.04	1	1000.00
9 3/5:APC	3.00	1918.41	2	1000.00
10 3/5:ATGM	0.0	0.78	0	1000.00
11 ATGM	0.0	1.76	0	1000.00
12 ATGM	0.0	0.38	0	1080.00
13 MORT 120	3.00	47.49	10	2000.00

Figure 12

SURVIVING FORCES AT TIME = 500. SECONDS:

BLUE GROUP	NUMBER	ROUNDS EXPENDED	TARGET
1 1B:TANK	0.19	28.92	1
2 1B:APC	2.59	1728.68	4
3 1B:RR 90	0.0	3.55	0
4 1B:RR106	0.0	1.68	0
5 2B:TANK	1.34	32.97	1
6 2B:APC	2.23	1808.32	4
7 2B:RR 90	0.0	3.44	0
8 3B:APC	4.00	2327.79	4
9 3B:RR 90	2.00	5.06	1
10 MORT 81	2.99	74.90	1
11 AFAAC 37	3.00	85.88	4

RED GROUP	NUMBER	ROUNDS EXPENDED	TARGET	DISTANCE FROM FEBA
1 4:TANK	0.39	26.93	1	0.0
2 4:APC	0.0	575.72	0	20.00
3 6:TANK	0.0	24.60	0	40.00
4 6:APC	3.89	1240.10	6	0.0
5 1/5:APC	3.00	2108.02	2	1050.00
6 1/5:ATGM	0.0	0.77	0	1050.00
7 2/5:APC	3.00	2624.79	6	1180.00
8 3/5:TANK	3.00	99.00	1	1000.00
9 3/5:APC	3.00	2018.52	2	1000.00
10 3/5:ATGM	0.0	0.78	0	1000.00
11 ATGM	0.0	1.76	0	1000.00
12 ATGM	0.0	0.38	0	1080.00
13 MORT 120	3.00	49.47	10	2000.00

***BREAK POINT: RED FORCES HAVE REACHED FEBA

Figure 13

PART E

INTELLIGENCE AND RECONNAISSANCE MODELS

In the general structure assumed to describe heterogeneous-force combat situations, the attrition coefficient was partitioned into three factors: the attrition rate, the allocation factor, and the intelligence factor. Prediction models for the attrition rate were developed in Part B of this report. Chapter 2 of Part D discussed the allocation factor in the context of optimal allocation strategies. This part of the report describes research to predict the form and parameters of the intelligence factor and also describes results of a small effort to model advance surveillance patrols associated with reconnaissance in force missions.

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Chapter 1

THE INTELLIGENCE COEFFICIENT

George Miller and Robert Farrell

This chapter describes the results of research to develop a predictive model of the intelligence factor, $I_{ij}(r)$, for the general heterogeneous-force combat model. Subscript notation for different groups and the dependency on range and time have been omitted for expository purposes.

Consider a renewal process consisting of occurrences of the event "a friendly unit stops firing and commences searching for another unit upon whom to fire." The process is depicted in Figure 1,

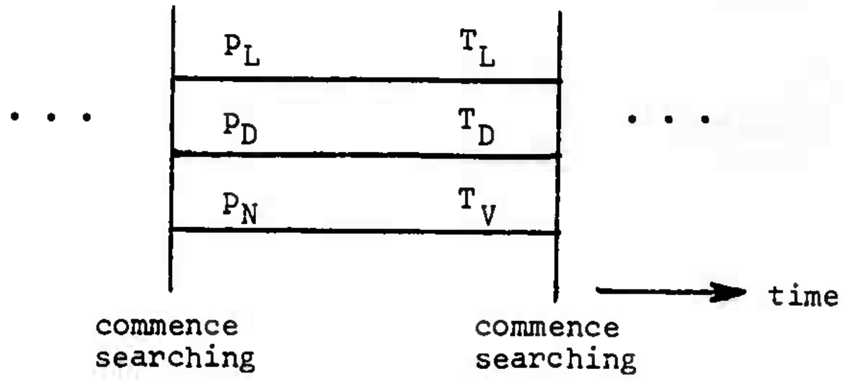

Figure 1. Renewal Search Process

where

T_L = the time between commencement of searches when a live target is acquired,

P_L = probability of acquiring a live target,

T_L = the time between commencement of searches when a dead target is acquired,

p_D = probability of acquiring a dead target,

T_V = the time between commencement of searches when an an area devoid of targets is acquired and mistaken for a target,

p_V = probability of acquiring an area devoid of targets and mistaking it for a target.

The times T_L, T_D, and T_V are random variables whose distributions have means $\overline{T}_L$, $\overline{T}_D$, and $\overline{T}_V$, respectively.[3] This formulation assumes that the actual search time is distributed throughout the firing time.

The expected time to accomplish one search and firing sequence is

$$p_L\overline{T}_L + p_D\overline{T}_D + p_V\overline{T}_V \quad .$$

[illegible] theorem (Parzen, 1962, p. 183) the mean number of search initiations in the interval [t, t + Δt] [illegible]

$$\frac{\Delta t}{p_L\overline{T}_L + p_D\overline{T}_D + p_V\overline{T}_V} \quad ,$$

[illegible] approaches infinity. Thus the mean number of kills in [illegible]

[illegible]

$$\frac{p_L \Delta \tau}{p_L \overline{T}_L + p_D \overline{T}_D + p_V \overline{T}_V} ,$$

where it is assumed that when a live target has been acquired the next search does not commence until the target has been killed. The attrition rate *including degradation due to imperfect surveillance* is

$$\frac{p_L}{p_L \overline{T}_L + p_D \overline{T}_D + p_V \overline{T}_V} .$$

Recalling that the allocation factor is unity for homogeneous-force battles, the attrition coefficient with imperfect intelligence was defined as the product of an attrition rate with perfect intelligence (α) and an intelligence coefficient (I). The attrition rate with perfect intelligence was shown in [B, 1.2] to be $1/\overline{T}_L$. Therefore,

$$\alpha I = \frac{1}{\overline{T}_L} I = \frac{p_L}{p_L \overline{T}_L + p_V \overline{T}_V + p_D \overline{T}_D} .$$

Accordingly, the intelligence coefficient is given by

$$I = \frac{p_L \overline{T}_L}{p_L \overline{T}_L + p_V \overline{T}_V + p_D \overline{T}_D} .$$

The formulation for I developed above assumes that, if a live target is acquired, it will be killed by the unit acquiring it before that unit stops firing. It seems reasonable to assume that there will be occasions when a unit will stop firing on a target before it is destroyed. This condition can be included in the formulation for I by letting

$\bar{T}_{L,K}$ = mean time between the commencement of searches when a live target is acquired and destroyed by the acquiring unit,

$P_{L,K}$ = probability of acquiring a live target and firing until it is destroyed,

$\bar{T}_{L,Q}$ = mean time between the commencement of searches when a live target is acquired but not killed by the acquiring unit,

$P_{L,Q}$ = probability of acquiring a live target and terminating attention to that target before it is destroyed,

$= P_L - P_{L,K}$.

Then considering the fundamental attrition rate (with perfect intelligence) as $1/\bar{T}_{L,K}$, the renewal process arguments yield

$$I = \frac{P_{L,K}\bar{T}_{L,K}}{P_{L,K}\bar{T}_{L,K} + P_{L,Q}\bar{T}_{L,Q} + P_D\bar{T}_D + P_V\bar{T}_V} .$$

The time $\bar{T}_{L,Q}$ could be expanded to distinguish between (a) stopping fire because the unit being fired upon was destroyed by another friendly unit, and (b) stopping fire due to mistakenly thinking the target had been destroyed.

1.1 References

Parzen, E., *Stochastic Processes*, San Francisco: Holden-Day, Inc., 1962.

Chapter 2

PRELIMINARY MODELING OF SURVEILLANCE PATROLS

Seth Bonder and Michael Moore

A small part of the intelligence research effort was devoted to the development of preliminary formulations of the reconnaissance in force mission. The purpose of this type of mission is to move and seize a distant objective. In doing so the force will use advance surveillance patrols to detect and avoid engaging the enemy while moving to the objective. The structures presented in previous parts of this report can be used to describe the assault phase on the objective. Efforts in the intelligence research focused on developing preliminary formulations of the surveillance or intelligence activity associated with the reconnaissance in force mission.

The development of structures to describe the surveillance activity require descriptions of (a) the behavior of the target in the environment, (b) the detection capability of the sensors used by the surviellance patrol, and (c) the interaction of these processes. This chapter describes simplified structures of these activities in which the target is deemed visible to the sensors for only single periods of time. Chapter 3 considers just the visibility process alone and develops more realistic descriptions of the target behavior as presented to the detection system.

2.1 General Situation

The surveillance situation examined is shown in Figure 1, where

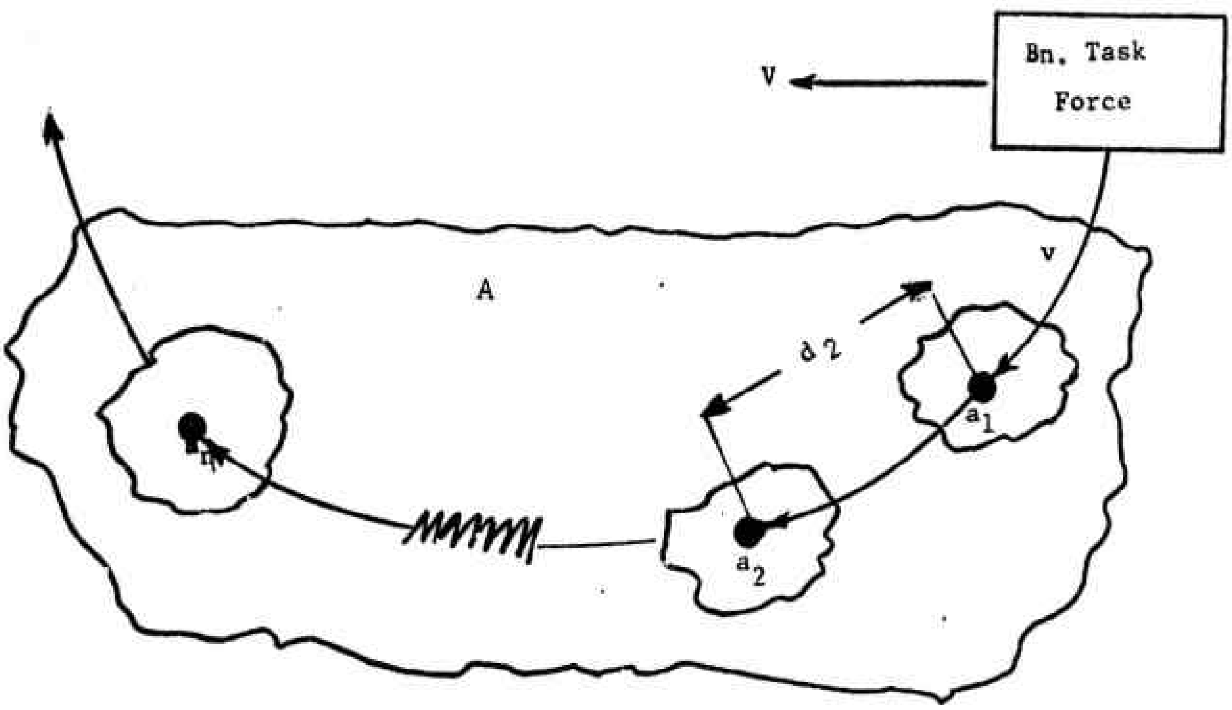

Figure 1. Surveillance Patrol

V = speed of main force,

v = speed of the surveillance patrol which advances to search the area A,

A = total area searched,

a_i = area of the i^{th} subarea searched (terrain dependent),

d_i = distance between subareas (i - 1) and i (terrain dependent). Assume that d_i = d for all i,

n = number of subareas searched.

Search in successive areas A is considered continuous search associated with the mobile force situation. Search in just one area A might be considered a periodic area surveillance to obtain general information during a static situation. For initial modeling purposes we assume that the reconnaissance patrol moves into a subarea and, as a unit, scans the area as a single sensor. We assume that the probability of detecting a target in the subarea is dependent on the presence of a target, its visibility, and the detection capabilities of the sensor. The patrol will leave the subarea at the time it detects the target's presence or after a specified time during which it has not detected a target. We initially consider the case in which there is only one target in A and assume

(1) Pr[target in A] = 1.0, and

(2) Pr[target in B < A] = B/A.

The target randomly moves from one locale to another in area A.

2.2 *Binary Visibility*

We consider first the situation in which the target and observer remain stationary while the latter scans the subarea for the former. In this case the two are either intervisible during the whole period the observer is scanning the i^{th} subarea or not intervisible at any time during the period. We define

τ_s = time spent in the subarea if a target is not detected,

τ_d = time required to detect a target when it is continuously visible to the sensor ($0 \leq \tau_d \leq \infty$),

p_v = probability that the target and observer are intervisible (intervisibility exists for total period),

$g(\tau)d\tau = \Pr[\tau \leq \tau_d \leq \tau + d\tau \mid \text{target present and visible}]$.

The density function $g(\tau)$ is a measure of sensor capabilities and would vary for different sensors. It has been obtained experimentally for visual sensing (Stollmack, 1968) and is also dependent on characteristics of the target and the environment. The probability p_v can also be obtained experimentally (see Nortronics, 1965) and is terrain dependent.[1]

The probability of detecting a target in the subarea is

$$p_i = \Pr[\text{target in subarea}] \cdot \Pr[\text{target visible for } \tau_s \mid \text{target present}] \cdot \Pr[\text{detect in } \tau_s \mid \text{target present and visible for } \tau_s]$$

$$= \left(\frac{a_i}{A}\right) p_v \int_0^{\tau_s} g(\tau)\, d\tau$$

$$= \frac{a_i}{A}\, p_v\, G(\tau_s) \qquad (1)$$

[1]Both the detection and visibility processes are, of course, range and search direction dependent; however, these dimensions are not be considered in this initial model.

Designating[2]

$$\tau_d^* = \text{time until the target is detected } (0 \le \tau_d^* \le \tau_s),$$

then

$$\Pr[\tau_d^* \le \tau] = \Pr[\tau_d \le \tau \,|\, \text{detection}]$$

$$= \Pr[\tau_d \le \tau \cdot \text{detection}]/\Pr[\text{detection}]$$

$$= \frac{\Pr[\tau_d \le \tau \cdot \text{target visible} \cdot \tau_d \le \tau_s]}{\Pr[\text{detection}]} . \qquad (2)$$

Since $\tau \le \tau_s$, the event $\tau_d \le \tau_s$ always occurs if the event $\tau_d \le \tau$ occurs. Therefore, (2) can be written

$$L(\tau) = \Pr[\tau_d^* \le \tau]$$

$$= \frac{\Pr[\tau_d \le \tau \cdot \text{target visible}]}{\Pr[\text{target visible} \cdot \tau_d \le \tau_s]}$$

$$= \frac{G(\tau)p_v}{p_v G(\tau_s)}$$

$$= \frac{G(\tau)}{G(\tau_s)} \qquad (3)$$

and

---

[2] The probability of target presence will be omitted from the following developments recognizing that it can be added in a straightforward manner.

$$\ell(\tau) = \Pr[\tau \leq \tau_d^* \leq \tau + d\tau]$$

$$= \frac{g(\tau)}{G(\tau_s)} \quad . \tag{4}$$

Letting

$$\tau_i = \text{time spent in } i^{th} \text{ area},$$

then

$$\Pr[\tau_i \leq \tau] = Y(\tau)$$

$$= \begin{cases} \Pr[\tau_d^* \leq \tau]\cdot\Pr[\text{detect target}] & \text{if } \tau < \tau_s \\ 1 & \text{if } \tau \geq \tau_s \end{cases}$$

$$= \begin{cases} L(\tau) \cdot p_i & \text{if } \tau < \tau_s \\ 1 & \text{if } \tau > \tau_s \end{cases} \tag{5}$$

$$= \begin{cases} G(\tau)p_v & \text{if } \tau < \tau_s \\ 1 & \text{if } \tau > \tau_s \end{cases} \tag{6}$$

and

$$y(\tau) = \begin{cases} \ell(\tau)p_i & \text{if } \tau < \tau_s \\ q_i & \text{if } \tau > \tau_s \end{cases} \tag{7}$$

where $q_i = 1 - p_i$. Graphically $Y(\tau)$ is shown in Figure 2.

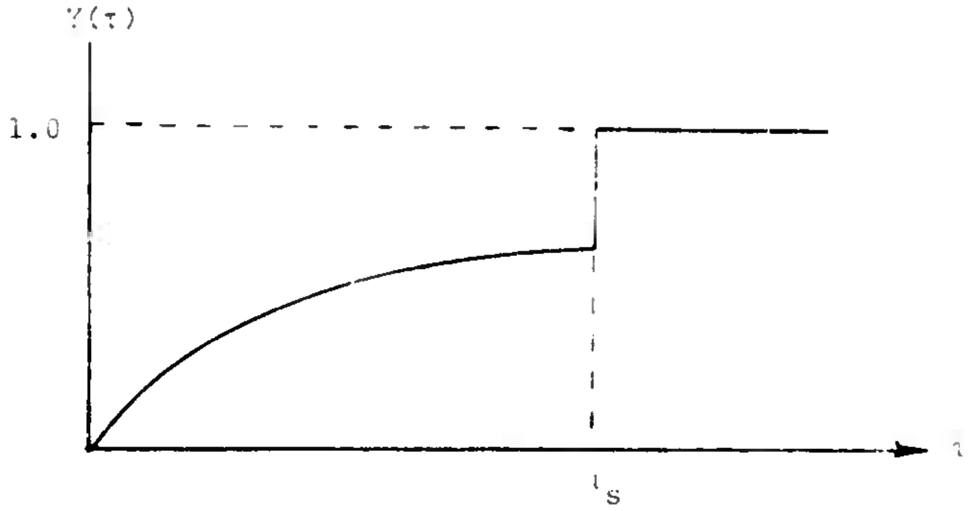

Figure 2. Distribution Function of τ_i

Consider the sequence of activities in which the reconnaissance patrol scans the first area, in which it may or may not detect the target, moves to the second area at speed v, scans the second area, moves to the third subarea, etc. We are interested in the random variable

$$T_1 = \text{time until the first detection.}$$

For a fixed number of trials, m, until the first detection:

$$T_1 = (m - 1)\tau_s + m(d/v) + \tau_d^*$$

$$= c_1 + c_2 m + \tau_d^* , \qquad (8)$$

where

$$c_1 = -\tau_s \qquad (9)$$

$$c_2 = d/v + \tau_s . \qquad (10)$$

[illegible] (8) and [illegible]

$$f_1(t|m)\,dt = \Pr[t \le T_1 \le t + dt|m], \tag{11}$$

then

$$f_1(t|m) = \ell(t - c_1 - c_2m)$$

$$= \frac{g(t - c_1 - c_2m)}{G(\tau_s)} \tag{12}$$

$$\text{for } (m-1)\,\tau_s \le t \le m\tau_s .$$

Consider first the possibility of searching more than n subareas. Then

$$f_1(t) = \sum_{m=1}^{\infty} f_1(t \cdot m)$$

$$= \sum_{m=1}^{\infty} f_1(t|m)\; s(m),$$

where

$s(m)$ = the probability of detecting the target on the m^{th} trial,

$= (1 - p)^{m-1}p,$

and $p_i = p$ for all i. Therefore, the probability of detecting the target before leaving the area is

$$\Pr[T_1 \le n\tau_s] = \int_0^{n\tau_s} f_1(t)\,dt . \tag{13}$$

The time to detect the target, given it is detected while the search is in process, is easily obtained if we note that

$$\Pr[\text{detect on trial } m \mid \text{detect on or before trial } n] = \frac{s(m)}{S(n)} ,$$

where

$$S(n) = \sum_{m=1}^{n} s(m) .$$

Therefore,

$$f_1(t \mid n) = \Pr[\text{time to detect} \mid \text{detection made within } n \text{ trials}]$$

$$= \sum_{m=1}^{n} f_1(t \mid m \cdot n) \frac{s(m)}{S(n)} . \qquad (14)$$

Consider next the random variable[1]

---

[1] The above model for p_i considered only one target uniformly distributed in A. Consideration of k uniformly distributed targets in A is obtained by using

$$p_k = 1 - (1 - a_i/A)^k$$

instead of (a_i/A) if it is assumed

(1) the subareas are chosen such that when more than one target is in the i^{th} subarea, they physically unite and appear as one target to the sensor;
(2) $g(\tau)$ reflects the increased size of the target; and
(3) p_v reflects the increased size of the target.

T_A = time spent searching the total area A.

For a fixed number of detections x,

$$T_A = \sum_{i=0}^{x} \tau^*_{d_i} + (n - x)\tau_s + n(d/v)$$

$$= \sum_{i=0}^{x} \tau^*_{d_i} + c_1 x + c_3, \qquad (15)$$

where $c_3 = n(\tau_s + d/v)$ and $\tau^*_{d_i}$ is the time to detect a target in the i^{th} subarea when a target is detected. Assuming that the $\tau^*_{d_i}$ are independent with pdf given by (4),

$$f_A{*}(t|x) = \Pr[t \leq T_A - c_1 x - c_3 \leq t + dt|x]$$

$$= \ell_1(\tau)*\ell_2(\tau)* \ldots *\ell_x(\tau), \qquad (16)$$

where the symbol * indicates the convolution of the densities. Since each of the $\ell_i(\tau)$ are truncated densities, the regions of integration must be carefully specified (as noted in the appendix to this chapter).

From (16) it follows that

$$f_A(t|x) = \Pr[t \leq T_A \leq t + dt|x]$$

$$= f_A{*}(t - c_1 x - c_3|x). \qquad (17)$$

Then

$$f_A(t) = \sum_{x=0}^{n} f_A(t\cdot x)$$

$$= \sum_{x=0}^{n} f_A(t|x)\phi(x), \tag{18}$$

where

$$\phi(x) = \binom{n}{x} p^x(1 - p)^{n-x} \quad . \tag{19}$$

It is important to note that the distributions for the time until the first detection, the time to search area A, etc., developed in equations 8-19 are solely dependent on knowing the distribution, $\ell(\tau)$, for the random variable τ_d^* .

2.3 Simple Interval Visibility: Time Zero to Random Limit

Consider next the situation in which the target or observer may move in the subarea such that intervisibility, if it exists, starts at time zero and lasts a random time period. Therefore, we consider the random variable

τ_v = time that the target remains visible

and let

$$h(\tau)\, d\tau = Pr[\tau \leq \tau_v \leq \tau + d\tau] .$$

Thus, rather than considering visibility of the target to

occur or not (with probability p_v) for the whole period $(0,\tau_s)$, we assume that a visibility time period of length τ_v can occur during the interval $(0,\tau_s)$ and that this visibility period begins at time zero. Then the probability of detecting a target in an infinitesimal interval $d\tau$ at time τ, $p_i(\tau)$ is

$$p_i(\tau) = \Pr[\tau \leq \tau_d \leq \tau + d\tau | \text{target present} \cdot \tau_v > \tau]\Pr[\tau_v > \tau]$$

$$= [g(\tau)\ d\tau]\left[\int_\tau^\infty h(\tau)\ d\tau\right] \tag{20}$$

and therefore the probability of detection is

$$p_i = \int_0^{\tau_s} g(\tau)\ \overline{H}(\tau)\ d\tau\ , \tag{21}$$

where

$$\overline{H}(\tau) = \int_\tau^\infty h(\tau)\ d\tau\ . \tag{22}$$

Continuing in an analogous fashion to Section 2.2, we have

$$\Pr[\tau \leq \tau_d^* \leq \tau + d\tau] = \frac{\Pr[\tau \leq \tau_d \leq \tau + d\tau | \tau_v > \tau]\Pr[\tau_v > \tau]}{\Pr[\text{detection}]}$$

or

$$\ell(\tau)\, d\tau = \frac{g(\tau)\, d\tau\, \overline{H}(\tau)}{P_i} \tag{23}$$

and

$$L(\tau) = \frac{\int_0^{\tau} g(\tau)\, \overline{H}(\tau)\, d\tau}{\int_0^{\tau_s} g(\tau)\, \overline{H}(\tau)\, d\tau} \,. \tag{24}$$

The probability of detecting the target is given by (13) and the distributions of the

(a) time until the first detection, given detection in n trials;

(b) time to search A given x detections occur; and

(c) time to search A

are given by (14), (17), and (18), respectively, when $\ell(\tau)$ of (23) is used in (12) and (16).

2.4 Simple Interval Visibility: Random Initiation and Limit

In the last section we considered the visibility to have a random time limitation but it was restricted to beginning at time zero. Suppose the visibility period starts at μ, where μ is a random variable with probability density function $f(\mu)$. Then the probability of detecting the target in an interval $d\tau$ at time τ, given that the visibility interval starts at time μ, is given by

$$p_i(\tau|\mu) = \begin{cases} Pr[(\tau-\mu) \leq \tau_d \leq (\tau-\mu) + d\tau \,|\, \text{target present} \\ \quad \cdot\ \tau_v > \tau-\mu] \cdot Pr(\tau_v > \tau-\mu) & \text{for } \tau \geq \mu \\ 0 & \text{for } \tau < \mu \end{cases}$$

and

$$p_i(\tau) = \int_0^{\tau_s} f(\mu) p_i(\tau|\mu)\, d\mu$$

$$p_i = \int_0^{\tau_s} f(\mu) \left[\int_0^{\tau_s} g(\tau-\mu)\, \overline{H}(\tau-\mu)\, d\tau \right] d\mu .$$

But the inside integrand is zero for $\tau < \mu$; hence,

$$p_i = \int_0^{\tau_s} f(\mu) \left[\int_\mu^{\tau_s} g(\tau-\mu)\, \overline{H}(\tau-\mu)\, d\tau \right] d\mu .$$

After a change of the variable of integration, one obtains

$$p_i = \int_0^{\tau} f(\mu) \left[\int_0^{\tau_s-\mu} g(t)\, \overline{H}(t)\, dt \right] d\mu . \qquad (25)$$

Continuing the analogy with Section 2.3,

$$L(\tau) = \frac{\int_{\mu=0}^{\tau} \int_{0}^{\tau_s} \frac{1}{\tau_s} [g(\tau-\mu)\, \bar{H}(\tau-\mu)\, d\tau]\, d\mu}{p_i} \tag{26}$$

and the relevant pdf's are obtained by employing the derivative of (26) in the appropriate equations as noted in that section.

2.5 *Areas for Future Research*

The methods described in this chapter provide a preliminary description of the surveillance activity associated with the reconnaissance in force mission. The methods suggest a number of possible extensions that should be examined with further research. These include

(1) consider the time limit τ_s as a random variable,

(2) consider multiple targets in the i^{th} subarea,

(3) remove assumption $p_i = p$,

(4) consider multiple sensors in i^{th} subarea,

(5) allow the subareas to overlap,

(6) include the range dimension since the input pdf's are range dependent,

(7) include the sensor search directi n and scan rate,

(8) interface the surveillance activity description with the differential models of the combat activity,

(9) determine "optimal" search strategies for the search sequence and time allocation for each subarea. Compare results to the classical search theories which do not consider the visibility process.

2.6 *References*

Nortronics Corporation, "Combat Vehicle Armament Study," AD 368-757, 1965.

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Appendix E, 2

CONVOLUTION OF TRUNCATED DENSITIES

Seth Bonder

Equation 16 of the text is a convolution of truncated distributions $\ell_i(\tau)$, $i = 1,2,\ldots,x$. For $x = 2$, the distribution of the sum is given by

$$f_2(t) = \begin{cases} \displaystyle\int_{\tau_2=0}^{t} \ell_1(t-\tau_2)\ell_2(\tau_2)\,d\tau_2 & 0 \le t \le \tau_s \quad (1.1) \\ \displaystyle\int_{\tau_2=t-\tau_s}^{\tau_s} \ell_1(t-\tau_2)\ell_2(\tau_2)\,d\tau_2 & \tau_s \le t \le 2\tau_s \quad (1.2). \end{cases} \qquad [1]$$

The regions of integration are shown in Figure A1. Equation

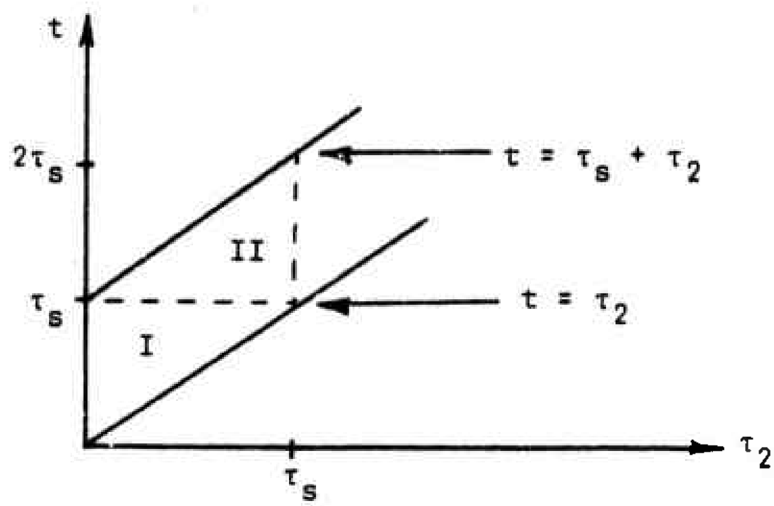


Figure A1. Convolution Region with x = 2

(1.1) is an integration over region I and (1.2) over region II.

For $x = 3$, the distribution of the sum $t = \tau_1 + \tau_2 + \tau_3$ is given by

$$
f_3(t) = \begin{cases}
\displaystyle\int_{\tau_3=0}^{t} f_2(t-\tau_3)\ell_3(\tau_3)\, d\tau_3 & 0 \le t \le \tau_s \quad (2.1) \\
\left.\begin{array}{l}\displaystyle\int_{\tau_3=t-\tau_s}^{\tau_s} f_2(t-\tau_3)\ell_3(\tau_3)\, d\tau_3 \quad (2.2a) \\ \displaystyle + \int_{\tau_3=0}^{t-\tau_s} f_2(t-\tau_3)\ell_3(\tau_3)\, d\tau_3 \quad (2.2b) \end{array}\right\} & \tau_s \le t \le 2\tau_s \quad (2.2) \\
\displaystyle\int_{\tau_3=t-2\tau_s}^{\tau_s} f_2(t-\tau_3)\ell_3(\tau_3)\, d\tau_3 & 2\tau_s \le t \le 3\tau_s. \quad (2.3)
\end{cases} \qquad [2]
$$

The regions of integration are shown in Figure A2.

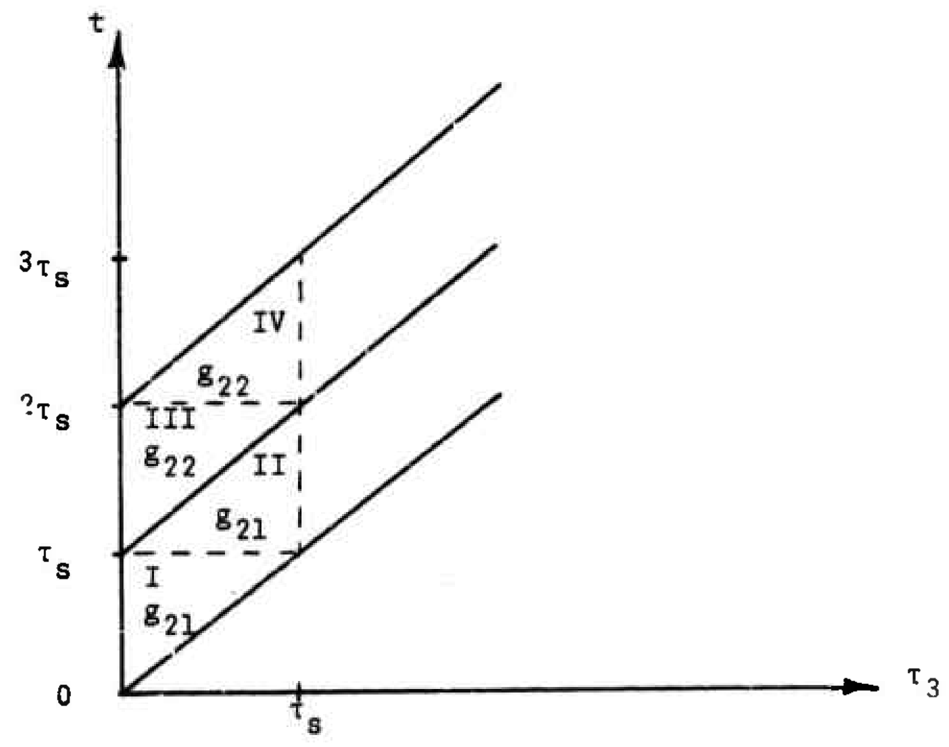


Figure A2. Convolution Region with x = 3

The terms in 2.1, 2.2a, 2.2b, and 2.3 are integrals over regions I, II, III, and IV, respectively. The $f_2(t)$ functions in [2] refer to different parts of [1] as follows:

In (2.1),	$f_2(t - \tau_2)$	$= g_{21}$	$= f_2(t - \tau_3)$	for $0 \leq t \leq \tau_s$
In (2.2a),	"	= "	"	" " " "
In (2.2b),	"	$= g_{22}$	$= f_2(t - \tau_3)$	" $\tau_s \leq t \leq 2\tau_s$
In (2.3),	"	= "	"	" " " " .

For $x = (n + 1)$ it is conjectured that the pdf of the sum is given by

$$f_{n+1}(t) = \begin{cases} \displaystyle\int_{\tau_{n+1}=0}^{t} f_n(t-\tau_{n+1})\ell_{n+1}(\tau_{n+1})\,d\tau_{n+1} & 0 \leq t \leq \tau_s \quad [n.1] \\ \left.\begin{array}{l} \displaystyle\int_{\tau_{n+1}=t-\tau_s}^{\tau_s} f_n(t-\tau_{n+1})\ell_{n+1}(\tau_{n+1})\,d\tau_{n+1} \quad (n.2a) \\ \displaystyle+\int_{\tau_{n+1}=0}^{t-\tau_s} f_n(t-\tau_{n+1})\ell_{n+1}(\tau_{n+1})\,d\tau_{n+1} \quad (n.2b) \end{array}\right\} & \tau_s \leq t \leq 2\tau_s \quad [n.2] \\ \cdots\cdots\cdots\cdots\cdots\cdots & \{n\} \\ \left.\begin{array}{l} \displaystyle\int_{\tau_{n+1}=t-(n-1)\tau_s}^{\tau_s} f_n(t-\tau_{n+1})\ell_{n+1}(\tau_{n+1})\,d\tau_{n+1} \quad [n.na] \\ \displaystyle+\int_{\tau_{n+1}=0}^{t-(n-1)\tau_s} f_n(t-\tau_{n+1})\ell_{n+1}(\tau_{n+1})\,d\tau_{n+1} \quad [n.nb] \end{array}\right\} & (n-1)\tau_s \leq t \leq n\tau_s \quad [n.n] \\ \displaystyle\int_{\tau_{n+1}=t-n\tau_s}^{\tau_s} f_n(t-\tau_{n+1})\ell_{n+1}(\tau_{n+1})\,d\tau_{n+1} & n\tau_s \leq t \leq (n+1)\tau_s \quad [n.(n+1)]. \end{cases}$$

The $f_n(t)$ functions in equation {n} refer to different parts of equation [n - 1] as shown in Figure A3 and below:

In (n.1), $f_n(t - \tau_{n+1}) = g_{n1} = f_n(t - \tau_{n+1})$ for $0 \leq t \leq \tau_s$

In (n.2a), " = " " " " " "

In (n.2b), " $= g_{n2} = f_n(t - \tau_{n+1})$ " $\tau_s \leq t \leq 2\tau_s$

In (n.3a), " = " " " " " "

In (n.3b), " $= g_{n3} = f_n(t - \tau_{n+1})$ " $2\tau_s \leq t \leq 3\tau_s$

. . . .

In (n.ja), " $= g_{nj-1} = f_n(t - \tau_{n+1})$ " $(j-2)\tau_s \leq t \leq (j-1)\tau_s$

In (n.jb), " $= g_{nj} = f_n(t - \tau_{n+1})$ " $(j-1)\tau_s \leq t \leq j\tau_s$

. . . .

In [n.na], " $= g_{n(n-1)} = f_n(t - \tau_{n+1})$ for $(n-2)\tau_s \leq t \leq (n-1)\tau_s$

In [n.nb], " $= g_{nn} = f_n(t - \tau_{n+1})$ for $(n-1)\tau_s \leq t \leq n\tau_s$

In [n.(n+1)], " = " " " " .

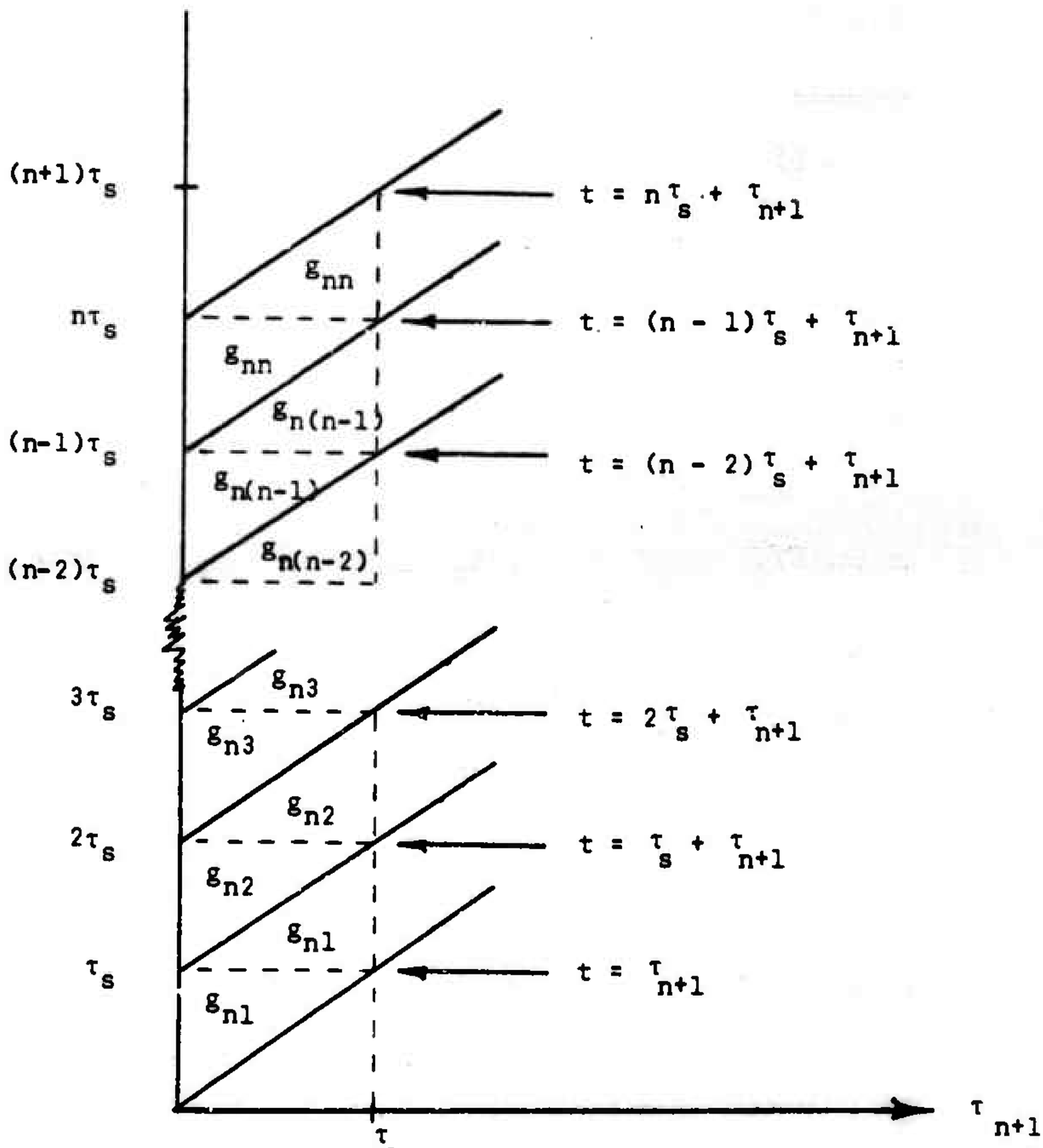


Figure A3. Convolution Region with x = n + 1

Chapter 3

A MULTIPLE INTERVAL VISIBILITY MODEL

Ralph Disney

In the previous chapter we examined simplified cases of the surveillance activity as part of the reconnaissance in force mission. The structures considered (a) imperfect characteristics of the sensing system as measured by the distribution $g(\tau)$, (b) simplified behavior of the target in terms of single periods of visibility to the sensing device, and (c) their interaction to determine the probability distributions of detecting targets. In this chapter we consider just the visibility process to develop a more realistic description of the target behavior in terms of multiple intervals of visibility. We are concerned with (among other things) the probability that a target is visible to the surveillance system's sensors at time t.[1] Specific questions of interest in this chapter are

(1) For a given t, what is the probability that the target is visible, $\Pi_1(t)$?

(2) Given that the target is visible at t, what is the length of time that he will remain visible?

(3) In t_s, how many times will he be visible?

(4) In t_s, what is the total time of visibility?

---

[1]In the simplest case where detection occurs with probability 1.0 if the target is visible, this is also the probability that the target is detected by the sensing device.

(5) If there are N independent targets, what is the probability density function for the number of visible targets at time t?

(6) If there are N targets, what is the probability density function for the number of sightings in $(0,t_s)$?

3.1 *The Visibility Model*

Consider first the one target case such that at $\tau = 0$, the target is visible (this is not a crucial assumption). At τ_1 the target becomes invisible. The length of $(0,\tau_1)$, say T_1, we assume to be a random variable with probability density function $f_1(t)$ and distribution function $F_1(t)$. The target remains invisible until τ_2 and the length of the interval (τ_1,τ_2) is a random variable T_2 with probability density $f_2(t)$ and distribution function $F_2(t)$. T_1 and T_2 are not necessarily identically distributed but are assumed to be independent. This description assumes that the target is alternately invisible, visible, invisible, etc. Succeeding lengths of time of visibility or invisibility are random variables, $T_1^{(n)}$, $T_2^{(n)}$, mutually independent, and the collection $\{T_1^{(n)}\}$ is independent of the collection $\{T_2^{(n)}\}$. The visibility periods are distributed as $T_1^{(n)}$. The invisibility periods are distributed as $T_2^{(n)}$. We could allow $T_1^{(1)}$ to be differently distributed than succeeding visibility periods but this generalization appears unnecessary at this time. Likewise we could allow all

of the $T_2^{(n)}$ to be differently distributed but there is nothing much to be gained except complex formulae. The crucial assumption for our present analysis is that the $T_i^{(n)}$'s are an independent collection of independent random variables. Even this assumption may be relaxed, but it will not be done here.

The mathematical structure that we impose on the target by the above assumptions is that of an alternating renewal process (Cox, 1962). Thus, we suppose that the target can be in one of two states with $X(t) = 1$ if the target is visible at t, and $X(t) = 2$ if the target is invisible at t. If $\{\tau_i: \ i = 1,2,\ldots\}$ are the times at which transitions occur, then we assume

$$\Pr[X(\tau_n) = 1 \mid X(\tau_{n-1}) = 2] = 1 \qquad n = 2,3,\ldots$$

$$\Pr[X(\tau_n) = 2 \mid X(\tau_{n-1}) = 1] = 1$$

$$\Pr[X(\tau_n) = i \mid X(\tau_{n-1}) = j] = 0 , \qquad \text{otherwise.}$$

In effect then, our target is behaving like a Markov chain over the states 1,2 when considered at the points of transition. (The process embedded at transition times is an embedded Markov chain). The one step transition matrix for this chain is

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In the simple case in which $f_1(t)$ and $f_2(t)$ are negatively exponentially distributed (n.e.d.), then $\{X(t)\}$ is also a Markov chain in continuous time and is completely analogous to the machine breakdown problem in queueing theory with one machine and one repairman. In the event $f_1(t)$ and $f_2(t)$ are not n.e.d., then $\{X(t)\}$ is not a Markov process (although $\{X(\tau_n)\}$ is). Thus, the initially interesting problems occur if we do *not* require $f_1(t)$ and $f_2(t)$ to be n.e.d. Because of the special structure of P, we call $\{X(t)\}$ an alternating renewal process.

A simple extension can consider several targets, each performing independently as an alternating renewal process. Let the stochastic process N(t) = the number of targets visible at time t. N(t) is a binomial process with $p = \Pi_1(t)$. Hence, analysis of this process should proceed in a straightforward manner. One advantage in introducing such a simple process is to allow an eventual extension to the case in which the targets are *not* operating independently.

3.2 Some Results from Renewal Theory

Some results from the theory of renewals will be needed to describe and study the visibility process. We summarize these briefly here. A full discussion is given by Cox (1962).

We introduce a renewal process as a sequence $\{X_n : n = 1,2,\ldots\}$ of independent, identically distributed non-negative random variables. The assumption of independence is crucial. The assumption of identical distributions is not crucial, but makes

developments easier. The non-negativity condition is certainly borne out in those cases of interest to detection.

It is assumed that each X_n is a continuous random variable with probability distribution function $Pr[X_n \leq t] = F(t)$ and corresponding density function $f(t)$.

The sequence $\{S_n : n = 1,2,...\}$ of partial sums,

$$S_n = X_1 + X_2 + \ldots + X_n \qquad n = 1,2,\ldots,$$

we call the "time until the n^{th} event."[1] Clearly, each S_n has a density function easily determined (in principle) from that of the X_n. Furthermore, $\{S_n : n = 1,2,...\}$ is a Markov process in discrete time with a continuous-state space as is seen by $S_n = S_{n-1} + X_n$, $n = 1,2,\ldots$. For fixed n we define

$$Pr[S_n \geq t] = H_n(t),$$

$H_o(t) = 1$ and the corresponding density $h_n(t)$.

The stochastic process $\{N(t);\ t \geq 0\}$ is called the counting process for the renewal process and represents the total number of events occurring in the fixed interval $(0,t)$. A fundamental identity for renewal processes is[2]

---

[1]For our later development it will be useful to include the event that occurs at zero in the count. Hence, if we formally require $X_o = 0$, as such, then S_n is really the time until the $(n+1)^{st}$ event.

[2]We have included the event at zero in these formulae.

$$N(t) \leq n \qquad \text{iff } S_n > t ,$$

from which one obtains

$$\Pr[N(t) \leq n] = \Pr[S_n > t] = 1 - H_n(t)$$

or

$$\Pr[N(t) = n] = H_{n-1}(t) - H_n(t) .$$

The expected number of events $E[N(t)] = M(t)$ is called the *renewal function* and can be shown to be given by

$$M(t) = \sum_{n=1}^{\infty} H_n(t) + 1 \quad .$$

The derivative $\frac{dM(t)}{dt} = m(t)$ is called the *renewal density function*. For well-behaved $H_n(t)$ it is easy to see that

$$m(t) = \sum_{n=1}^{\infty} h_n(t) . \qquad (1)$$

A critically important observation here motivates our later work. Although $m(t)$ is defined as the derivative of $M(t)$, and $m(t)$ can be obtained from (1) for given $f(t)$, $m(t)$ can also be considered as a *probability*. In particular, it is observed from (1) that for a given interval $(t, t + \Delta t)$, $h_n(t)\Delta t$ is the probability that the time until the n^{th} event occurs has a length of $(t, t + \Delta t)$. In another sense, $h_n(t)\Delta t$

is the probability that the n^{th} event occurs in the interval $(t, t + \Delta t)$. From (1), it then follows that $m(t)\Delta t$ is the probability that some (perhaps the first or second or third...) event, after the one at 0, occurs in the interval $(t, t + \Delta t)$. Thus, $m(t)\Delta t$ can be interpreted as the probability that some renewal occurs in $(t, t + \Delta t)$. For $\Delta t \to 0$ one interprets $m(t)dt$ as "the probability that an event occurs at t" (more precisely, the probability that an event occurs in $(t, t + dt)$.

3.3 Application to the Visibility Process

With the background developed so far, it is clear that the visibility process $\{X(t)\}$ is an alternating renewal process and that there exists a body of knowledge which can be applied directly.

Figure 1 represents one realization of the $\{X(t)\}$ process

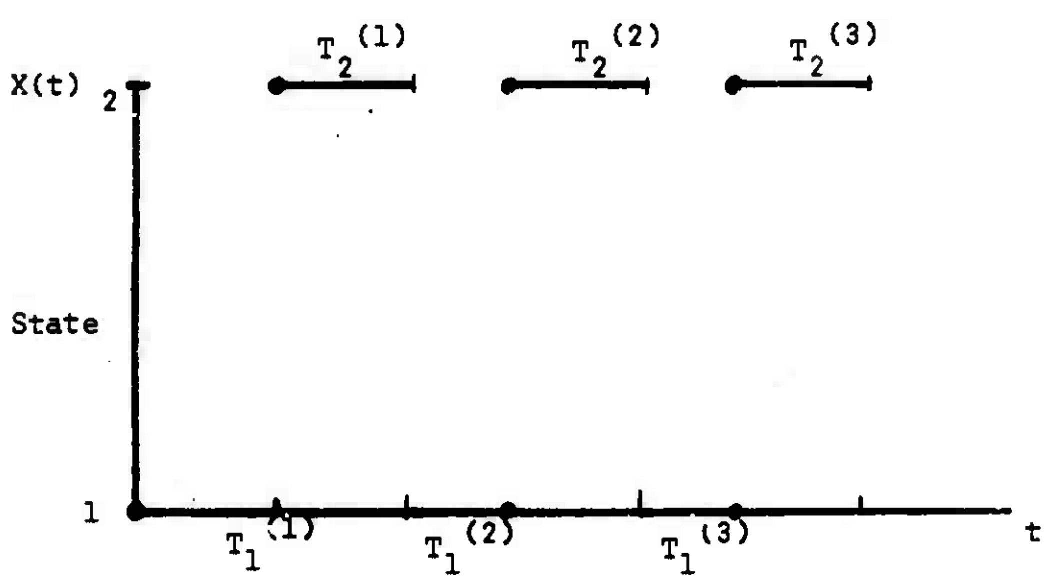


Figure 1   A Realization of the $\{X(t)\}$ Process

We define

$$Y_1 = T_1^{(1)} + T_2^{(1)}$$

$$Y_2 = T_1^{(2)} + T_2^{(2)} .$$

And in general

$$Y_n = T_1^{(n)} + T_2^{(n)} .$$

The sequence $\{Y_n : n = 1,2,\ldots\}$ is a sequence of independent, identically distributed, non-negative random variables, or $\{Y_n\}$ is a renewal process. The independence condition follows since the $T_i^{(n)}$ are all mutually independent and the Y_n are formed as nonoverlapping sums. Hence, one can define a new stochastic process $\{Y_n : n = 1,2,\ldots\}$, a renewal process, whose Y_n are identically distributed as

$$g_{Y_n}(t) = f_1(t) * f_2(t)$$

where * designates convolution of the densities.

In a similar manner let

$$Z_1 = T_1^{(1)}$$

$$Z_2 = T_2^{(1)} + T_1^{(2)}$$

$$Z_3 = T_2^{(2)} + T_1^{(3)}$$

and in general

$$Z_n = T_2^{(n-1)} + T_1^{(n)}.$$

Then $\{Z_n\}$ is also a renewal process with "lifetime" distributed as

$$\ell_{Z_n}(t) = f_1(t) * f_2(t) \qquad \text{for } n = 2,3,\ldots,$$

and

$$\ell_{Z_1}(t) = f_1(t) .$$

Thus, $\{Z_n\}$ has Z_1 differently distributed than the other Z_n. Except for this, $\{Z_n\}$ is an ordinary renewal process. Renewal processes with Z_1 differently distributed than Z_n, $n = 2,3,\ldots$ are called delayed renewal processes.

It is interesting to note that $\{Y_n\}$ is a renewal process, $\{Z_n\}$ is a renewal process, but $\{Z_n\}$ is not independent of $\{Y_n\}$. For a given n, Z_{n+1} and Y_n depend on each other through $T_2^{(n)}$.

For each of the processes $\{Y_n\}$, $\{Z_n\}$ all of the results for renewal theory are valid (for the Z_n process one must adjust the formulae to account for the differences in the distribution of Z_1 and Z_n but this adjustment is trivial).

In particular, one has for fixed n

$$_YS_n = Y_1 + Y_2 + \ldots + Y_n \quad ,$$

$$_ZS_n = Z_1 + Z_2 + \ldots + Z_n \quad .$$

$\{_YS_n\}$ is a stochastic process for the Y-process, which for the problem at hand is the time until the n^{th} invisibility ceases. Reinterpreted $_YS_n$ is a random variable giving the time until the $(n + 1)^{st}$ appearance of the target. Similarly, $_ZS_n$ is a random variable giving the time until the n^{th} disappearance of the target.

If we define $N_Y(t)$ = the number of occurrences of events in the Y-process, then $N_Y(t)$ is the number of times the target is invisible in $(0,t)$. In particular $N_Y(t_s)$ is the number of times the target will be invisible in $(0,t_s)$. From the fundamental identity of renewal theory[1]

$$N_Y(t) \leq n \qquad \text{iff } {}_YS_n > t$$

and

$$\Pr[N_Y(t) \leq n] = \Pr[{}_YS_n > t]$$

$$\Pr[N_Y(t) = n] = \begin{cases} H_{n-1} - H_n, & n = 1,2,3,\ldots \\ H_0 \overset{\text{def}}{=} 1, & \end{cases}$$

---

[1] We are counting the occurrence at 0 in this process.

which in principle can be used to determine the probability density function for the number of times the target is invisible in (0,t).

In exactly analogous fashion,

$$Pr[N_Z(t) < n] = Pr[{}_Z S_n > t]$$

can be used to determine the number of times the target is visible in (0,t).[1]

Similarly, $M_Y(t)$ = the mean number of invisible periods in (0,t), the renewal function for the Y-process; $M_Z(t)$ = the renewal function for the Z-process (visibili- -). In principle, all of these values can be determined at this point.

We would like to determine the probability that the target is visible at t, $\Pi_1(t)$, for interaction with the detection process. That is, we would like to determine the state probability $Pr[X(t) = 1] = \Pi_1(t)$. With the above information, this probability is easily obtained. Clearly, X(t) = 1 in two (and only two) mutually exclusive cases

[1]Here we do not count an event at 0.

Case 1. The system is in state 1 at t because it was in state 1 at 0 and never left.

Case 2. The system is in state 1 at t because it became visible in (u, u + du) and remained visible for a time t - u or more.

The probabilities of these two cases are simple to determine. For case 1 to occur, the initial visibility period T_1 has to be of length t or more. The probability of this is simply

$$Pr[T_1 > t] = 1 - F_1(t) = F_1^{(c)}(t) .$$

Case 2 requires an argument but the facts are at hand. For the target to be visible at t, having entered the visibility state (state 1) in (u, u + du), there had to be an occurrence of an event in the Y-process in (u, u + du). From our discussion of the renewal density function, $m_Y(u)$ du is the probability of an event in the Y-process occurring in (u, u + du). Now this event in the Y-process put the system into state 1 (every epoch in the Y-process is an entrance to state 1). For the target to be in the state 1 (visible) at t, this entrance to state 1 must have occurred and the target must occupy state 1 at least from u to t or for a length of time (t - u). The probability that the system enters state 1 at u and remains in that state for a length of time (t - u) is simply $m_Y(u)[1 - F_1(t - u)]du$. Since we are not concerned with when the target entered state 1

only that it be in state 1 at t , the probability for case 2 is

$$\int_0^t m_Y(u)\ F_1^{(c)}(t - u)du\ .$$

Hence, in total

$$\Pi_1(t) = F_1^{(c)}(t) + \int_0^t m_Y(u)\ F_1^{(c)}(t - u)du$$

$$\Pi_2(t) = 1 - \Pi_1(t)\ .$$

Thus, one has, in principle, the probability that the target is visible at time t, $\Pi_1(t)$. $\Pi_2(t)$ is the probability that the target is not visible at t.

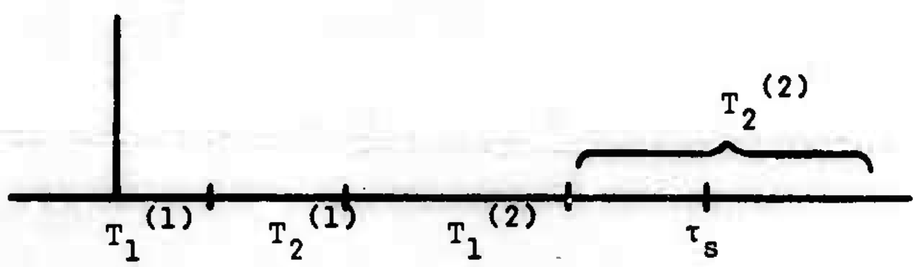
If we assume that the detection system is always observing the area of the target and will, with probability 1.0, and see the target if it is visible, then $\Pi_1(t)$ is the probability that the target will be detected at t. This may be the j^{th} time it has been detected. We have already discussed the number of sightings in (0,t).

If there are N targets in the area, each independently acting as an alternating renewal process, then the number of targets visible at t will be a binomially distributed random variables with parameters $\Pi_1(t)$ and N.

A random variable of interest to interact with an imperfect detection process is T = the total time that the target is visible

in $(0,\tau_s)$ for some fixed value τ_s. To determine the probabilities for T, one must consider two cases, illustrated below.

Case 1: τ_s occurs during an invisibility period.



For a fixed value of $N_Y(\tau_s)$, say $N_Y(\tau_s) = h$,

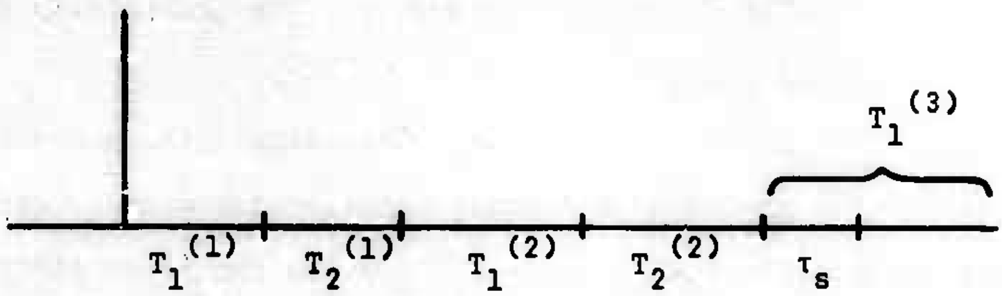
$$T = T_1^{(1)} + T_1^{(2)} + \ldots + T_1^{(h)} .$$

Thus it is clear that T is first the sum of h, independent, identically distributed r.v. and the density function of T is thus the h-fold convolution of $f_1(t)$ with itself. Let $f_T(t) = f_1(t)^{*h}$, h fixed, denote this convolution. But this is for h fixed. Hence the joint density of T and N_Y is $f_T(t)$. $\Pr[N_Y\tau_s) = h] = f_1(t)^{*h} \Pr[N_Y(\tau_s) = h]$. The marginal density

$$\Pr[t \leq T \leq t + dt] = g_T(t) = \sum_{h=1}^{\infty} f_1(t)^{*h} \Pr[N_Y(\tau_s) = h]$$

where T = the total time spent in the visible state and τ_s is fixed. The probability that case 1 occurs is $\Pi_2(\tau_s)$.

Case 2: τ_s occurs during a visibility period.



Defining T as before, one can proceed as before. However, T is now made up of h - 1 random variables, T_1 and one random variable, U, which is not a T_1 (it is the backward recurrence time of a T_1). U is not a particularly appealing random variable so we seek another way to find $g_T(t)$ for this case. Define I = the total length of invisibility time in $(0,\tau_s)$. Then clearly I can play the role of T in case 1. Hence, the unconditional density of I can be found exactly as in case 1. The argument applies from there with only a change of $f_2(t)$ for $f_1(t)$ and $N_Z(\tau_s)$ for $N_Y(\tau_s)$. Now, obviously, for fixed τ_s,

$$T = \tau_s - I .$$

Hence,

$$\Pr[T \leq t] = 1 - \Pr[I \leq \tau_s - t] .$$

Since the density function of I can be found, the density function

for T can be found for this case easily. The probability that case 2 occurs is $\Pi_1(\tau_s)$.

The total probability for T is then simply the probability mixture of the probability density function for case 1 [with mixing probability $\Pi_2(\tau_s) = 1 - \Pi_1(\tau_s)$] and the probability density function for T from case 2 [with mixing probability $\Pi_1(\tau_s)$.]. This density function is valid for $t < \tau_s$.

For $t = \tau_s$, one sees immediately that $Pr[T = \tau_s] = F_1^{(c)}(\tau_s)$. Clearly, $Pr[T > \tau_s] = 0$.

Thus to summarize: let T = total length of time the target is visible in $(0,t_s)$. Then,

$$Pr[T = \tau_s] = 1 - F_1(\tau_s)$$

$$Pr[T \leq t] = \Pi_2(\tau_s) \int_0^t \sum_{h=1}^{\infty} f_1(t)^{*h} Pr[N_Y(\tau_s) = h]dt$$

$$+ \Pi_1(\tau_s) \left\{ 1 - \int_0^{\tau_s - t} \sum_{h=1}^{\infty} f_2(t)^{*h} Pr[N_Z(\tau_s) = h]dt \right\}$$

$$\text{if } t < t_s$$

$$Pr[T \leq t] = 1 , \quad \text{if } t > t_s.$$

3.4 *References*

Cox, D.R., *Renewal Theory*, New York: John Wiley and Sons, Inc., 1962.

PART F

MISCELLANEOUS RESEARCH AREAS

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This part of the report presents results of research effort categorized as "miscellaneous research areas." Each area is so classified since (a) it may not be related to the mainstream of the project research as differential models of combat, or (b) only a small amount of study effort was devoted to the topic, thus, the analysis is not carried out in complete detail or merely sketches the direction of analysis.

Chapter 1 describes research which adds the dimension of reliability and provides a structure for adding mobility to "one-on-one" stochastic duel theory. Chapter 2 considers optimal assault speeds for the variable attrition-rate, homogeneous-force battle model to maximize the ratio of survivors and also sketches a power series solution for a linear, heterogeneous-force, differential battle model. Analysis of ammunition requirements in context of differential models is considered in Chapter 3.

Chapter 1

RELIABILITY AND MOBILITY IN THE THEORY OF STOCHASTIC DUELS

David Thompson

Principal efforts in the research program have been devoted to extending the macroscopic differential theories of combat to include mobility of both forces, microscopic details of weapon systems in predicting the attrition rates, and the fact that the attrition rates vary when forces employ mobile weapon systems. This approach was taken based on the judgment that it would be more difficult at this time to enrich the stochastic duel theories (which already considered microscopic weapon system parameters) to include more than single duelists and simultaneously consider mobility of these forces. A small effort, however, was devoted to extending the "one-on-one" stochastic duel descriptions to include reliability of the duelist's weapons and initial elements of mobility. The results of this research are presented in this chapter following a brief review of existing stochastic duel theories.

1.1 *Survey of the Theory of Stochastic Duels*

Although the theory has been extended under rather restrictive assumptions to treat engagements involving many weapons, the "one-on-one" duel has received the most treatment. The term

"fundamental duel" is used in the literature to denote a one-on-one duel in which each weapon has constant single-shot kill probabilities, unlimited ammunition, unlimited time to complete the duel and a projectile with negligible time-of-flight. The time between a contestant's rounds are independent, identically distributed random variables with both participants' firing processes being initiated simultaneously. The following outline gives an overview of published results in stochastic duel theory.

One-Versus-One-Duels

A: Fundamental duel. Williams and Ancker (1963a) give the distribution of the time until a kill for a marksman firing at a passive target and give the win probability for a fundamental duel. Some variants of the fundamental duel are also given--the classical duel in which both sides fire at time zero, a duel in which one contestant gets a free shot at time zero, and a duel in which the element of surprise is probabilistic. In a second article Williams and Ancker (1965a) developed the win probability for a fundamental duel with constant firing times. Groves (1964) also treated a fundamental duel with constant firing times, obtaining the probability that a side has survived the n^{th} round, while similar survival models were treated by Schoderbek (1962a, 1962b).

B. Time duration. Ancker (1966a) added to the fundamental duel the possibility of a time limit ending the duel in a tie if both opponents are still alive. Time limits which are both

fixed and random variables are treated, as are both constant and random firing-times. Williams and Ancker (1964a) later found the distribution and the first three moments of the time duration of the fundamental and the time-limited duel.

C. Ammunition. Ancker (1964a) gave a general solution for fundamental duels in which each side has a limited initial amount of ammunition which is either a constant or a random variable. Ancker and Gafarian (1964b) found the distribution and the first two moments of the number of rounds fired in a fundamental and an ammunition-limited duel.

D. Time-of-flight. Ancker (1966b) added fixed and continuous random-projectile flight times to the fundamental duel. A delay procedure is treated in which a duelist waits to observe the effects of a round before the next round is fired, as is a no-delay procedure in which there is no such wait. Time-of-flight, limited initial ammunition, and ammunition replenishment were all added to the fundamental duel by Jaiswal and Bhashyam (1966). They treated random time-of-flight with a no-delay firing procedure, assumed a fixed initial amount of ammunition on each side, and allowed fixed ammunition replenishment at negative exponentially distributed times.

E. Round-dependent hit probabilities. The possibility of a marksman improving his accuracy on successive rounds was considered by Williams (1964b, 1965b) in a fundamental duel with negative exponential firing times and with hit probabilities

following a Pascal distribution of their position in the firing sequence. Bhashyam and Singh (1967) treated a similar duel with negative exponential firing times and hit probabilities which are general functions of the number of rounds fired, but with the added feature of each side starting the duel with a specified amount of ammunition, either finite or infinite.

F. Approximations. Williams (1963b) obtained an approximation to the win probability in the fundamental stochastic duel in terms of the single-shot kill probabilities and the first two moments of the distributions of the firing-time distributions.

G. Markov firing doctrine. In the models treated so far in this survey each participant's firing times were either constant or were a series of independent, identically distributed, continuous random variables. Chapters 2 and 3, Part B, of this report describe methods of predicting the time-to-kill probability distributions for systems that employ the Markov firing doctrine in which the time between rounds and the hit probabilities (after the first round) can each take on different values, depending on whether the preceding round hit or missed the target. These distributions can be used in a standard manner to obtain the win probabilities for weapons which employ the Markov firing doctrine.

Many-Versus-Many Duels

A. Triangular and square duels. Williams and Ancker (1965a) treated three cluster duels characterized by

simultaneous, fixed-time firings and common hit probabilities on each side. One duel is a two-versus-one triangular duel, and the others are two-versus-two square duel. In one square duel the contestants are paired off until one is killed, at which time the opponents concentrate their fire on the remaining weapon. In the other square duel one side concentrates its fire on one of the opponents. A comparison of the two duels indicated that concentrating its fire lessened a side's chances of winning.

B. M-versus-N duels. Similar M-versus-N attrition models with common hit probabilities and fixed firing times on each side were treated by Robertson (1956) and Helmbold (1966), who gave the probability that a given number of targets survive a given number of rounds. Various methods of assigning weapons to targets and approximations to survival probabilities were considered. Schoderbek (1962b) did some similar work with random times between rounds. Helmbold (1968) incorporated a different attrition situation into a duel. The two sides trade a number of simultaneous volleys, but at each volley each surviving unit fires at a prearranged opponent, independently of whether or not it still survives.

C. Relations to differential models. Williams (1963b) obtained approximate solutions to two M-versus-N duels in which the number of contestants was large and all weapons on a side fired simultaneously at negative exponentially distributed intervals with common kill probabilities. When all duelists

could fire on any opposing unit at all times, the solution was the classical Lanchester square law, and when only individual combats were allowed, Lancaster's linear law was obtained. Robertson (1956) also mentions the connection of the M-versus-N duel to Lanchester's laws.

Research Outline

Ancker (1964a, 1966a) included some natural limitations of weapon systems in duels involving limited ammunition supplies and time limits. Another natural limitation of a weapon is the reliability of its firepower. The denigration of a weapon may be due to factors such as severe natural environment, lack of preventive maintenance, and use of the weapon when fired. The first two factors concern the study of reliability and maintenance per se, while the third factor is more complex, since more than the temporary loss of firepower is at stake in combat. Sections 1.2 and 1.3 of this chapter address catastrophic failures of firepower, leaving the duelist entirely helpless or forcing him to withdraw from the duel. Reliability is treated both as a function of time and as a function of the number of rounds fired, the latter as a more realistic model which relates the chance of breakdown to actual use of the system. The probability of one side winning is found for all the duels, and the results are compared with those for the corresponding fundamental duel. Some connections with stochastic duels found in the literature are noted.

Ancker (1964c) notes that mobility has not been treated in stochastic duels with the exception of some very simple situations. Section 1.4 of this chapter treats stochastic duels with time-dependent single-shot kill probabilities, the main interpretation of this model being the dependence of hit probabilities upon range and range varying with time. The treatments of mobility in a duel is a new development, although a general attrition model with some similarities is treated by Farrell (1968a, 1968b, 1968c).

Unless otherwise stated, the basic assumptions of the fundamental duel are employed in this chapter. These assumptions for each duelist are

(i) The hit probability is a constant;

(ii) The time between any two successive rounds follows the same probability distribution;

(iii) "Hit probability" is synonymous with "kill probability"; so the effect of a hit is not cumulative.

1.2 *Stochastic Duels Involving Continuous Reliability Processes*

Two weapon systems, A and B, have firepower subsystems which are subject to mechanical failure independent of vulnerability to the opposing weapon. The lifetime of each firepower subsystem is a continuous-valued random variable with known p.d.f. A failure is catastrophic and cannot be repaired on the battlefield. If a failed weapon is capable of withdrawing from the duel, there are two possible times at which

the withdrawal can be made. The failure may be detected at the instant it occurs, and the retreat made then, or the failure may not be detected until an attempt is made to fire the next round and the firepower subsystem is found to be inoperable. Let

L_A = lifetime of A's firepower subsystem,

L_B = lifetime of B's firepower subsystem,

$r_A(t)$ = p.d.f. of A's lifetime,

$r_B(t)$ = p.d.f. of B's lifetime,

$R_A(t) = Pr\{L_A > t\}$,

$R_B(t) = Pr\{L_B > t\}$.

1.2.1 No Withdrawal Case

In this case a failed weapon remains in the duel, which is terminated only by that weapon's destruction or by the opponent's failure. Since the firing sequences of the two duelists are independent, as are their reliability processes, the uncoupling property used to study the fundamental duel can be employed. By the uncoupling property the duel can be regarded as equivalent to a marksmanship contest in which the participants fire independently at separate targets. The first duelist to destroy his target wins the duel, but if both contestants fail before a kill occurs, the duel is concluded as a tie.

Each duelist's firing sequence is independent of his reliability process until a breakdown occurs in his firepower subsystem. No more rounds can be fired after a breakdown.

With the independence in mind, the following symbols will be defined for a marksman subject to no breakdowns. Let

p_A = A's single-shot kill probability,

p_B = B's single-shot kill probability;

$f_A(t)$ = p.d.f. of the time between A's rounds;

$f_B(t)$ = p.d.f. of the time between B's rounds;

T_A = time for A to destroy a passive target, given he is free from failures;

T_B = time for B to destroy a passive target, given he is free from failures.

If we let

$$h_A(t)\, dt = Pr\{t \leq T_A < t + dt\} \quad ,$$

then the p.d.f. of T_A is

$$h_A(t) = \sum_{n=1}^{\infty} p_A q_A^{\,n-1} f_A^{*n}(t) \quad ,$$

where $q_A = 1 - p_A$ and $f_A^{*n}(t)$ is the n-fold convolution of $f_A(t)$ with itself (Williams and Ancker, 1963a). The density $h_B(t)$ is defined in the same way.

Each marksman is subject to failures. Let

$$g_B(t)dt = Pr\{B \text{ kills his target in the interval } (t, t + dt)\}$$

$$g_B(t)dt = Pr\{L_B \geq t \cdot t \leq T_B < t + dt\}$$

Then $g_B(t) = h_B(t)R_B(t)$ since B can kill at t only if his lifetime is greater than t. The probability that B has not destroyed his target by t is

$$1 - \int_0^t h_B(x) R_B(x)\, dx \quad . \qquad (1)$$

Then the probability that A wins is the probability that A destroys his target before A fails and before B destroys his target. The probability A wins is

$$P(A) = \int_0^\infty h_A(t) R_A(t) \left[1 - \int_0^t h_B(x) R_B(x)\, dx \right] dt \qquad (2)$$

Since withdrawal from this duel is impossible for a failed weapon, a tie can occur only if both weapons fail before a kill has occurred. The tie probability is

$$P(AB) = \Pr\{L_A < T_A \cdot L_B < T_B\} \quad ,$$

and since the behavior of the two marksmen is independent,

$$P(AB) = \Pr\{L_A < T_A\} \cdot \Pr\{L_B < T_B\} \quad ,$$

and

$$F(AB) = \int_0^\infty r_A(t) \left[\int_t^\infty h_A(x) dx \right] dt \quad \int_0^\infty r_B(t) \left[\int_t^\infty h_B(x) dx \right] dt \qquad (3)$$

As an example, consider that the times between rounds and failure times are all negative exponentially distributed. Then

$$f_A(t) = \gamma_A e^{-\gamma_A t} \qquad f_B(t) = \gamma_B e^{-\gamma_B t}$$

$$r_A(t) = \lambda_A e^{-\lambda_A t} \qquad r_B(t) = \lambda_B e^{-\lambda_B t}$$

Since

$$f_A^{*n}(t) = \frac{\gamma_A(\gamma_A t)^{n-1} e^{-\gamma_A t}}{(n-1)!} ,$$

then
$$h_A(t) = p_A\gamma_A \exp(-p_A\gamma_A t)$$

$$h_B(t) = p_B\gamma_B \exp(-p_B\gamma_B t) .$$

The probability that B has not killed his target by t, obtained by substituting into (1), is

$$\left(\frac{\lambda_B}{p_B\gamma_B + \lambda_B}\right) + \left(\frac{p_B\gamma_B}{p_B\gamma_B + \lambda_B}\right) \exp[-(p_B\gamma_B + \lambda_B)t]$$

From (2)

$$P(A) = \frac{p_A\gamma_A[\lambda_B(p_A\gamma_A + \lambda_A + p_B\gamma_B + \lambda_B) + p_B\gamma_B(p_A\gamma_A + \lambda_A)]}{(p_A\gamma_A + \lambda_A)(p_B\gamma_B + \lambda_B)(p_A\gamma_A + p_B\gamma_B + \lambda_B)} .$$

When $\lambda_A = 0$ and $\lambda_B = 0$,

$$P(A) = \frac{p_A\gamma_A}{p_A\gamma_A + p_B\gamma_B}$$

the result obtained by Williams and Ancker (1963a) for the case of exponential firing times without considering reliability. Using (3), the probability of a tie is

$$P(AB) = \frac{\lambda_A\lambda_B}{(p_A\gamma_A + \lambda_A)(p_B\gamma_B + \lambda_B)} ,$$

which is zero when breakdowns are impossible. If P(B) is found by a formula symmetric to that for P(A), it can be shown that P(A) + P(B) + P(AB) = 1.

1.2.2 Withdrawals at Failure Points

In this case a weapon is removed from action at the instant it breaks down, ending the duel in a tie if no kill has yet occurred. Side A will win the duel if his time-to-kill is less than B's time-to-kill and less than both lifetimes of the firepower systems.

$$P(A) = Pr\{T_A < T_B \cdot T_A < L_B \cdot T_A < L_A\}$$
$$= \int_0^\infty h_A(t)\left[\int_t^\infty h_B(x)dx\right] R_B(t)R_A(t)\, dt \quad . \tag{4}$$

Since a duelist may leave the duel when his weapon fails, a tie occurs when either weapon fails before a kill has occurred. The tie event is composed of two disjoint events:

$$P(AB) = Pr\{L_A < T_A \cdot L_A < T_B \cdot L_A < L_B\}$$
$$+ Pr\{L_B < T_A \cdot L_B < T_B \cdot L_B < L_A\}$$

$$= \int_0^\infty r_A(t)R_B(t)\int_t^\infty h_A(x)dx \int_t^\infty h_B(x)dx\, dt$$

$$+ \int_0^\infty r_B(t)R_A(t)\int_t^\infty h_A(x)dx\int_t^\infty h_B(x)dx\, dt\,. \quad (5)$$

In order to show that in general $P(A) + P(B) + P(AB) = 1$ the following notation will be used: Let

$$H_A(t) = \int_t^\infty h_A(x)dx$$

$$H_B(t) = \int_t^\infty h_B(x)dx\,.$$

From (4) and (5), and the fact that $P(B)$ may be obtained by the same steps as $P(A)$,

$$P(A) + P(B) + P(AB) = \int_0^\infty r_A(t)R_B(t)H_A(t)H_B(t)\, dt$$

$$+ \int_0^\infty r_B(t)R_A(t)H_A(t)H_B(t)\, dt$$

$$+ \int_0^\infty h_A(t)R_A(t)R_B(t)H_B(t)\, dt$$

$$+ \int_0^\infty h_B(t)R_A(t)R_B(t)H_A(t)\, dt$$

$$= -\int_0^\infty H_A(t)H_B(t)d[R_A(t)R_B(t)]$$

$$-\int_0^\infty R_A(t)R_B(t)d[H_A(t)H_B(t)]$$

$$= -\int_0^\infty d[H_A(t)H_B(t)R_A(t)R_B(t)] = 1.$$

The last step obtains since $r_A(t)$, $r_B(t)$, $h_A(t)$, and $h_B(t)$ are probability density functions.

As an example, again consider all the densities to be negative exponential. Equation 4 reduces to

$$P(A) = \frac{p_A\gamma_A}{p_A\gamma_A + \lambda_A + p_B\gamma_B + \lambda_B}$$

and (5) yields

$$P(AB) = \frac{\lambda_A + \lambda_B}{p_A\gamma_A + \lambda_A + p_B\gamma_B + \lambda_B} .$$

When $\lambda_A + \lambda_B = 0$, P(A) becomes the solution for a fundamental duel with n.e.d. firing times.

The duel which permits withdrawal can be considered as a duel with a random time limit. The time limit occurs when either A or B suffers a breakdown. Let L(t) be the distribution function of the time limit. Then

$$1 - L(t) = Pr\{L_A > t\} \cdot Pr\{L_B > t\} ,$$

since the reliability processes are independent, and L(t) becomes

$$L(t) = 1 - R_A(t)R_B(t) \quad .$$

Either by differentiating L(t) or by a probabilistic argument the density of the time limit can be found to be

$$\ell(t) = r_A(t)R_B(t) + r_B(t)R_A(t) \ .$$

Stochastic duels with random time limits have been treated by Ancker (1966a), where he also treats fixed firing times and fixed time limits. The duel treated above can be viewed as a duel in which each participant has his own random time limit. The results agree with those of Ancker when negative exponential reliabilities and firing times are used with the above formula for $\ell(t)$.

1.2.3 Withdrawals at Firing Points

Consider again the assumption that a failed weapon can withdraw from the duel, but in this case the failure is not discovered until the first attempt to fire a round following the failure. Let

T_A = A's time to destroy a passive target,

T_B = B's time to destroy a passive target,

L_A = lifetime of A's firepower subsystem,

L_B = lifetime of B's firepower subsystem,

L_A^* = time A detects his failure,

L_B^* = time B detects his failure.

Note that (i) $L_A \leq L_A^*$ and (ii) L_A^* and T_A are dependent such that A kills his opponent only if $T_A < L_A^*$. The probability that A wins is simply

$$P(A) = \Pr\{T_A < T_B \cdot T_A < L_A^* \cdot T_A < L_B^*\} .$$

Since the firing sequences of the two contestants as marksmen are dependent,

$$P(A) = \int_0^\infty w_A(t) \cdot \Pr\{T_B > t \cdot L_B^* > t\}dt , \qquad (6)$$

where

$$w_A(t)\, dt = \Pr\{t \leq T_A < (t + dt) \cdot L_A^* > t\}$$

$$= \Pr\{t \leq T_A < (t + dt) \cdot L_A > t\}$$

and

$$w_A(t) = h_A(t) \int_t^\infty r_A(x)\, dx \quad . \qquad (7)$$

The probability that B has not killed his target and not discovered a failure by time t is

$$\Pr\{T_B > t \cdot L_B^* > t\} = \int_0^t \left[\int_x^\infty r_B(y)dy \cdot \int_{t-x}^\infty f_B(y)dy \cdot \sum_{n=1}^\infty q_B^n f_B^{*n}(x)\right] dx$$

$$+ \int_t^\infty f_B(x)\, dx \quad . \qquad (8)$$

Assuming all the p.d.f.'s are negative exponential, (7) becomes

$$w_A(t) = p_A\gamma_A \exp[-(p_A\gamma_A + \lambda_A)t] \quad .$$

From (8)

$$\Pr\{T_B > t \cdot L_B^* > t\} = \exp(-\gamma_B t)$$

$$+ q_B\gamma_B \exp(-\gamma_B t) \int_0^t \exp[(q_B\gamma_B - \lambda_B)x]dx \; .$$

Two possibilities exist in evaluating (8). If $q_B\gamma_B \neq \lambda_B$,

$$\Pr\{T_B > t \cdot L_B^* > t\} = \left(\frac{q_B\gamma_B}{q_B\gamma_B - \lambda_B}\right) \exp[-(p_B\gamma_B + \lambda_B)t]$$

$$- \left(\frac{\lambda_B}{q_B\gamma_B - \lambda_B}\right) \exp(-\gamma_B t) \quad ,$$

and if $q_B\gamma_B = \lambda_B$,

$$\Pr\{T_B > t \cdot L_B^* > t\} = (q_B\gamma_B t + 1) \exp(-\gamma_B t) \; .$$

From (6) the probability that A wins is

$$P(A) = \frac{p_A\gamma_A(p_A\gamma_A + \lambda_A + \gamma_B + \lambda_B)}{(p_A\gamma_A + \lambda_A + \gamma_B)(p_A\gamma_A + \lambda_A + p_B\gamma_B + \lambda_B)}$$

for both the cases. For n.e.d. firing times, P(A) reduces to

the solution for the fundamental duel.

Since the formula for P(B) is symmetric to that for P(A), the tie probability is

$$P(AB) = 1 - P(A) - P(B)$$

$$= \frac{\lambda_A(\lambda_B+\gamma_A)(p_A\gamma_A+\lambda_A+\gamma_B) + \lambda_B(\lambda_A+\gamma_B)(p_B\gamma_B+\lambda_B+\gamma_A)}{(p_A\gamma_A+\lambda_A+\gamma_B)(p_B\gamma_B+\lambda_B+\gamma_A)(p_A\gamma_A+\lambda_A+p_B\gamma_B+\lambda_B)} .$$

1.2.4 The Effect of Reliability on Duel Results

It is possible that a weapon's chances of failure during a duel are so insignificant that a model including reliability processes need not be used to predict the duel outcome. One criterion for deciding whether there is a significant difference between the fundamental duel and the duel including reliability is the relative error in the win probability. Let

$P_f(A)$ = P(A) for a fundamental duel,

$P_r(A)$ = P(A) for a fundamental duel with reliability processes included.

Then an α significant difference between the models exists if

$$\frac{P_f(A) - P_r(A)}{P_r(A)} > \alpha$$

for some arbitrary α. For the n.e.d. failure and firing times and the withdrawal case given in section 1.2.2, this condition

becomes

$$\frac{\lambda_A + \lambda_B}{p_A\gamma_A + p_B\gamma_B} > \alpha \ .$$

This expression indicates that an α significant difference exists if the ratio of the sum of the failure rates to the sum of the attrition rates exceeds a specified α value.

If the error is measured relative to $P_f(A)$ rather than $P_r(A)$, the condition becomes

$$\frac{P_f(A) - P_r(A)}{P_f(A)} > \alpha \quad .$$

For the preceding example this reduces to

$$\Pr(AB) > \alpha \ ,$$

where Pr(AB) is the tie probability for the *reliability* model.

This analysis suggests that there is little need to include reliability in models of many battlefield situations, such as tank duels, which are usually characterized by high attrition rates. However, there are some situations in which reliability can be an important factor. One class of situations is that characterized by abnormally high failure rates, which may occur when a weapon is in a harsh natural environment or has been in the field for a long time without preventive maintenance. Another class of situations is characterized by low attrition rates, which occur when well-entrenched positions

cause low single-shot kill probabilities.

Two possibilities were considered when a weapon was allowed to withdraw from the duel at the discovery of a firepower failure. In Section 1.2.2 retreat was possible at the instant of the failure. The probability A wins in this case will be called P_1. In Section 1.2.3 a breakdown was not discovered until an attempt was made to fire the next round, at which time retreat was possible. The probability A wins in this case will be called P_2. It can be shown that

$$P_2 = P_1 \left(\frac{p_A \gamma_A + \lambda_A + \gamma_B + \lambda_B}{p_A \gamma_A + \lambda_A + \gamma_B} \right).$$

The correction factor in parentheses can range from one to infinity, indicating that Section 1.2.2 underrates A's chances if the assumptions of Section 1.2.3 hold. One might conjecture that in practice λ_B will be small compared to $p_A \gamma_A + \lambda_A + \gamma_B$, so that the use of the simpler model of Section 1.2.2 would suffice under both sets of assumptions.

1.3 Stochastic Duels Involving Round-Dependent Failures

A weapon may be such that failures occur only at those instants at which rounds are fired and that the probability of failure on any round is a function of its position in the firing sequence. The function could vary from duel to duel for any weapon because of varying usage and preventive maintenance.

This type of reliability assumption relates the chance of failure to actual usage of the weapon.

The number of the round on which the failure occurs is a random variable and since the number of rounds fired by a weapon is one less than the number of the failure round, the number of rounds allotted to a weapon in a duel is also a random variable. This fact means that the reliability duel is similar to a duel involving limited ammunition, in which the number of rounds a weapon has at the start of the duel is a random variable.

1.3.1 No Withdrawal Case

In this case the duel is continued until one of the opponents is destroyed or both duelists suffer firepower failures. Let

$$\begin{aligned} \alpha_k &= P_r\{A \text{ fails on round } k + 1\} \\ &= P_r\{A \text{ starts with } k \text{ rounds of ammunition}\} \\ \beta_j &= P_r\{B \text{ fails on round } j + 1\} \\ &= P_r\{B \text{ starts with } j \text{ rounds of ammunition}\}, \end{aligned}$$

where

$$\sum_{k=0}^{\infty} \alpha_k = 1$$

and

$$\sum_{j=0}^{\infty} \beta_j = 1 \quad .$$

Since the failure probability can be interpreted as the probability of being allotted k-1 rounds of ammunition, the reliability-limited duel has the same mathematical form as a duel with a random allotment of ammunition. This duel ends only when one duelist is killed or both run out of ammunition. Ancker (1964a) obtained a solution for the random ammunition-limited duel without withdrawal. By defining

$$h_{A1}(t) = P_A \sum_{k=1}^{\infty} \alpha_k \sum_{\ell=0}^{k-1} q_A^{\ell} f_A^{*\ell}(t) \quad ,$$

$h_{B1}(t)$ similarly, and by letting

$$G_B(t) = \int_t^{\infty} h_{B1}(x)dx + \sum_{k=0}^{\infty} \alpha_k q_A^k \quad ,$$

the win and tie probabilities are found by Ancker to be

$$P(A) = \int_0^{\infty} G_B(t) h_{A1}(t)dt$$

$$P(AB) = \sum_{k=0}^{\infty} \alpha_k q_A^k \sum_{j=0}^{\infty} \beta_j q_B^j \quad .$$

1.3.2 Withdrawal Case

In this case we assume that a failed weapon can withdraw from the duel without delay, ending the duel in a tie. This duel is similar to an ammunition-limited duel in which retreat is permitted. A participant is allotted N rounds at the start of the duel and fires all N rounds if possible. In a duel

studied by Ancker (1964a) the duelist withdraws at the instant he fires the N^{th} round. An alternative, however, is that the duelist does not know that the N^{th} round is the last. He will attempt to fire the N + 1st round and withdraw when he finds that his weapon does not fire. If we take the probability of being allotted N rounds as the probability of a firepower failure at the N + 1st round, this duel is seen to be equivalent to a duel with reliability included. This situation has not been treated in the literature.

Ideally, the probability of breakdown at a firing point should be a general probability function of the number of rounds fired, but for the sake of achieving a mathematically tractable model, the probability of a failure will be a constant, independent of the number of preceding rounds. As before, the times between rounds are continuous, independent random variables, and the duel is stationary in the sense that the single-shot kill probabilities remain constant. Let

p_A = Pr{A's round fires and hits the target},

p_B = Pr{B's round fires and hits the target},

q_A = Pr{A's round fires and misses the target},

q_B = Pr{B's round fires and misses the target},

u_A = Pr{A's round fails},

u_B = Pr{B's round fails},

where

$$p_A + q_A + u_A = 1$$

and

$$p_B + q_B + u_B = 1 .$$

The number of rounds a side is allotted to fire at a passive target is geometrically distributed. The probability that a firepower breakdown occurs on round n+1 is

$$\Pr\{A \text{ has a supply of } n \text{ rounds}\} = u_A(1 - u_A)^n.$$

This two opponent duel can be considered as a marksmanship contest. A and B start at the same time firing at passive targets. The one who destroys his target first wins the duel if his opponent has not as yet retreated. A can win the duel at time t only if he hits the target at t, he has not failed or hit the target before t, B has not hit his target yet, and B has not failed before t. The probability that A kills his target in the interval (t, t + dt) is given by

$$h_A(t)dt = \sum_{n=1}^{\infty} p_A q_A^{n-1} f_A^{*n}(t)dt ,$$

where f_A^{*n} is the n-fold convolution of A's firing-time distribution with itself. That is, for some n, his n^{th} shot hits the target at t with none of the preceding n-1 rounds hitting or failing to fire.

The probability that B has arrived at time t without killing his target or suffering a failure of his firepower system is

$$G_B(t) = \Pr\{B \text{ fired no shots in } (0, t)\}$$
$$+ \Pr\{B \text{ fired one shot in } (0,t), \text{ which missed}\}$$
$$+ \Pr\{B \text{ fired two shots in } (0,t), \text{ which missed}\}$$
$$+ \ldots.$$

The probability that B fired no shots in $(0,t)$ is

$$\int_t^\infty f_B(x)\,dx \ .$$

The probability that exactly n rounds were fired in $(0,t)$ is

$$\int_0^t f_B^{*n}(x)\left[\int_{t-x}^\infty f_B(y)\,dy\right]dx \ ,$$

where the integral in brackets indicates that the n + 1st round has not been fired by time t and where f_B^{*n} is the n-fold convolution of B's firing-time distributuion. Since the probability that all n rounds missed the target without causing a firepower failure is q_B^n, the probability that B is operative at t and has not destroyed his target is

$$G_B(t) = \int_t^\infty f_B(x)dx$$
$$+ \sum_{n=1}^\infty q_B^n \int_0^t f_B^{*n}(x)\left[\int_{t-x}^\infty f_B(y)dy\right]dx \ .$$

The win probability is the integral over all t of the probability that A is able to effect a win at t,

$$P(A) = \int_0^\infty h_A(t) G_B(t)\, dt \ .$$

A tie happens when either side fails before a kill has occurred. This event is the union of two mutually exclusive events, that A fails before B fails or a kill occurs and that B fails before A fails or a kill occurs. We consider first the probability that A fails before B fails or a kill occurs. The probability that A fails in the interval $(t, t + dt)$ without having hit his target is

$$w_A(t)dt = \sum_{n=1}^{\infty} u_A q_A^{n-1} f_A^{*n}(t)\, dt \quad ,$$

and the probability that B has neither hit his target nor broken down by time t is just $G_B(t)$. The probability that A fails first and causes a tie is then

$$\int_0^\infty w_A(t) G_B(t) dt \quad .$$

By a similar argument let

$$w_B(t)dt = \sum_{n=1}^{\infty} u_B q_B^{n-1} f_B^{*n}(t) dt$$

and

$$G_A(t) = \int_t^\infty f_A(x)dx + \sum_{n=1}^{\infty} q_A^n \left\{ \int_0^t f_A^{*n}(x) \left[\int_{t-x}^{\infty} f_A(y)dy \right] dx \right\} .$$

The tie probability is

$$P(AB) = \int_0^\infty w_A(t)G_B(t)dt + \int_0^\infty w_B(t)G_A(t)dt \quad .$$

As an example, let the firing-time distributions be negative exponential.

$$f_A(t) = \gamma_A e^{-\gamma_A t} \qquad f_B(t) = \gamma_B e^{-\gamma_B t} .$$

Using the fact that

$$f_A^{*n}(t) = \frac{\gamma_A(\gamma_A t)^{n-1} e^{-\gamma_A t}}{(n - 1)!} ,$$

and similarly for $f_B(t)$, it can be shown that

$$h_A(t)dt = p_A \gamma_A e^{-(p_A + u_A)\gamma_A t} dt$$

and

$$G_B(t) = e^{-(p_B + u_B)\gamma_B t} .$$

The win probability is then

$$P(A) = \frac{p_A \gamma_A}{(p_A + u_A)\gamma_A + (p_B + u_B)\gamma_B} \quad .$$

For $u_A = 0$ and $u_B = 0$, P(A) becomes the solution to the fundamental duel with n.e.d. firing times.

Substituting into the equations for the tie probability,

$$w_A(t) = u_A\gamma_A e^{-(p_A+u_A)\gamma_A t}$$

$$w_B(t) = u_B\gamma_B e^{-(p_B+u_B)\gamma_B t}$$

$$G_A(t) = e^{-(p_B+u_B)\gamma_B t}$$

and

$$P(AB) = \frac{u_A\gamma_A + u_B\gamma_B}{(p_A + u_A)\gamma_A + (p_B + u_B)\gamma_B} \, .$$

Developing P(B) in the same way P(A) is found, it can be shown that P(A) + P(B) + P(AB) = 1.

1.4 A Duel with Time-Dependent Hit Probabilities

The literature of the theory of stochastic duels has dealt almost exclusively with combat situations which are static with respect to mobility. In all models in the literature a weapon's single-shot kill probability is either a constant or a function of the round's position in the firing sequence. The latter situation is most simply interpreted as "homing," the improvement in accuracy a marksman makes in successive shots at a target. Neither constant nor round-dependent kill probabilities are adequate to model a dynamic duel situation in which the distance between the opponents varies, and their

single-shot kill probabilities subsequently vary with time. A simple extension of the one-on-one is described in this section in which time-dependent kill probabilities are considered, and the win probability determined.

The duel situation is a fundamental duel in which the firing-times are negative exponentially distributed, but the hit probabilities are allowed to be continuous, integrable functions of time bounded by zero and one. The problem of a marksman firing at a passive target will be treated first, then the uncoupling property will be used to obtain the duel win probability.

Let

$$\gamma\Delta t = \Pr\{\text{one round is fired in } (t, t + \Delta t)\},$$

$$p(t) = \Pr\{\text{a round fired at time t destroys the target}\},$$

$$G(t) = \Pr\{\text{the target is alive at time } t\}.$$

It should be remembered that $1/\gamma$ is the mean time between rounds. The probability of more than one round being fired in t is just $0(\Delta t)$, where

$$0(\Delta t) = \text{terms of the order of } \Delta t, \text{ such that}$$

$$0(\Delta t)/\Delta t \to 0 \text{ as } \Delta t \to 0$$

From the above definitions, classical arguments lead to

$$\begin{aligned} G(x + \Delta x) &= G(x)[1 - \gamma \Delta x] + G(x)\gamma\Delta x[1 - p(x)] + 0(\Delta t) \\ &= G(x)[1 - \gamma p(x)\, \Delta x] + 0(\Delta t)\ . \end{aligned}$$

By subtracting G(x) and taking the limit as x approaches zero, the derivative of G(x) is found to be

$$dG(x)/dx = -\gamma p(x)G(x) . \tag{9}$$

Rearranging (9) and integrating both sides from zero to t yields

$$\int_0^t \frac{dG(x)}{G(x)} = -\gamma \int_0^t p(x)\,dx \quad .$$

Since G(0) = 1, the left-hand side can be written simply as lnG(t) - lnG(0) = lnG(t), and G(t) can be written as

$$G(t) = \exp\left[-\gamma \int_0^t p(x)dx\right] . \tag{10}$$

Now let h(t)dt be the probability that the marksman effects his kill in the interval (t, t + dt). If G(t) → 0 as t → ∞ , h(t) is the probability density of the time to effect a kill. Using the fact that the probability of a kill by time t is 1 - G(t), h(t) = -dG(t)/dt. Then by substituting (10) into (9),

$$h(t) = \gamma p(t) \exp\left[-\gamma \int_0^t p(x)dx\right] . \tag{11}$$

A duel between two sides, A and B, can be considered as a marksmanship contest in which the first side to destroy his target is the winner. The probability A wins is

$$P(A) = \int_0^\infty G_B(t) h_A(t)\, dt \ , \tag{12}$$

where $G_B(t)$ is the probability B's target is alive at t, and $h_A(t)dt$ is the probability A effects a kill in the interval $(t, t + dt)$. If the firing rates and hit probabilities are appropriately subscripted, P(A) becomes

$$P(A) = \int_0^\infty \gamma_A p_A(t) \exp\left[-\gamma_A \int_0^t p_A(x)dx - \gamma_B \int_0^t p_B(x)dx\right] dt \ .$$

Similarly, the probability B wins is

$$P(B) = \int_0^\infty G_A(t) h_B(t)\, dt \ .$$

Using the fact that $h_A(t)dt = -dG_A(t)$ and $h_B(t)dt = -dG_B(t)$,

$$P(A) + P(B) = G_A(0)G_B(0) - \lim_{x \to \infty} G_A(x)G_B(x) \ .$$

Since $G_A(0) = 1$ and $G_B(0) = 1$, P(A) and P(B) will sum to one if, and only if, $G_A(x)$ or $G_B(x)$ approaches zero as x approaches infinity. Each of these events will occur if $\int_0^\infty p_A(x)\, dx$ or $\int_0^\infty p_B(x)\, dx$ is unbounded. The probability of a tie is

$$P(AB) = \lim_{t \to \infty} G_A(t)G_B(t) \ ,$$

which is zero when the probability of a kill is a certainty for either A or B.

If $p_A(t) = p_A$ and $p_B(t) = p_B$, constants for all t, then from (10) and (11), $G_B(t) = \exp[-p_B\gamma_B t]$ and $h_A(t) = p_A\gamma_A \exp[-p_A\gamma_A t]$ and (12) gives $P(A) = p_A\gamma_A/(p_A\gamma_A + p_B\gamma_B)$. This is the result of the classic fundamental duel.

1.4.1 A Closing Engagement

Consider a duel in which the duelists start at a given distance r_s, close at a constant speed v for a given time t_o, and then halt and finish the duel if necessary. It is assumed that for the distances involved, the single-shot kill probabilities are inversely proportional to the square of the distance between the duelists. The distance between the duelists at any time t is

$$r = r_s + vt\ , \quad 0 \le t \le t_o$$

$$r = r_s + vt_o\ , \quad t_o \le t$$

since v = dr/dt is negative. Employing the above assumption for distances between $r_s + vt_o$ and r_s, the single-shot kill probabilities are $p_A = a/r^2$ and $p_B = b/r^2$. As functions of time the kill probabilities are

$$p_A(t) = \begin{cases} \dfrac{a}{(r_s + vt)^2}\ , & 0 \le t \le t_o \\[2ex] \dfrac{a}{(r_s + vt_o)^2}\ , & t_o \le t \end{cases}$$

$$p_B(t) = \begin{cases} \dfrac{b}{(r_s + vt)^2}, & 0 \leq t \leq t_o \\ \dfrac{b}{(r_s + vt_o)^2}, & t_o \leq t . \end{cases}$$

The integrals of the hit probabilities with respect to time are

$$\int_0^t p_A(x)dx = \begin{cases} \dfrac{at}{r_s(r_s + vt)}, & 0 \leq t \leq t_o \\ A_1 + p_A(t_o)t, & t_o \leq t \end{cases}$$

$$\int_0^t p_B(x)dx = \begin{cases} \dfrac{bt}{r_s(r_s + vt)}, & 0 \leq t \leq t_o \\ B_1 + p_B(t_o)t, & t_o \leq t \end{cases},$$

where

$$A_1 = avt_o^2/(r_s + vt_o)^2$$

and

$$B_1 = bvt_o^2/(r_s + vt_o)^2 .$$

The integration to find P(A) can be done in two parts.

$$\int_0^{t_o} G_B(x)h_A(x)dx = \frac{a\gamma_A}{a\gamma_A + b\gamma_B}\left[1 - \exp\left[-\frac{(a\gamma_A + b\gamma_B)t_o}{r_s(r_s + vt_o)}\right]\right].$$

$$\int_{t_o}^{\infty} G_B(x)h_A(x)dx = \frac{a\gamma_A}{a\gamma_A + b\gamma_B}\exp\left[-\frac{(a\gamma_A + b\gamma_B)t_o}{r_s(r_s + vt_o)}\right] .$$

Then

$$P(A) = a\gamma_A/(a\gamma_A + b\gamma_B)$$

It is interesting to note that P(A) is independent of the initial distance of the duelists, the attack speed, and the final distance. This is due to the fact that the kill probabilities are always in a constant ratio and each marksman is certain to kill his target given enough time. In fact P(A) always equals $a\gamma_A/(a\gamma_A + b\gamma_B)$ under the following conditions:

$$p_A(t) = ag(t) \quad a \neq 0$$

and

$$p_B(t) = bg(t) \quad b \neq 0 \ ,$$

where

$$0 \leq g(t) \leq \max(1/a,\ 1/b)$$

and

$$\int_0^t g(x)dx \to \infty \quad \text{as } t \to \infty \ .$$

1.4.2 Flight Time and Mobility

Stochastic duels with flight times were considered by Ancker and Gafarian (1966b) and structurally includes the effect of mobility. Projectile flight times which vary

linearly with time were added to the fundamental duel, which could describe an engagement in which forces are closing or separating at a constant speed. Ancker's model was a very special case of a dynamic engagement, since the kill probabilities of the participants were constants over time. A more realistic situation would be to allow flight time and kill probability both to change with distance. This is considered below, using Ancker's assumption of linear flight times.

The situation consists of two participants separating or closing at a constant speed, then halting and finishing the duel. If τ is the flight time for side A, then

$m = d\tau/dt$ a constant,

$b = \tau$ at initial posiiton in duel,

$a = \tau$ when duelists halt.

The time-of-flight for A's projectiles is then

$$\tau = b + mt \quad \text{for } 0 \le t \le (a - b)/m$$

$$\tau = a \quad \text{for } (a - b)/m < t \ .$$

The probability A wins is given by Ancker as

$$P(A) = \int_0^{\frac{a-b}{m}} h_A(t) \left[\int_{(1+m)t+b}^{\infty} h_B(x)dx \right] dt$$

$$+ \int_{\frac{a-b}{m}}^{\infty} h_A(t) \left[\int_{t+a}^{\infty} h_B(x)dx \right] dt , \qquad (13)$$

where $h_A(t)$ is the p.d.f. of A's time-to-kill and $h_B(t)$ is the p.d.f. of B's time-to-kill. Both functions are p.d.f.'s since the hit probabilities are constants after the duelist halts and a kill is a certainty for each participant given enough time. Mobility is taken into account in the probabilities if the kill-time densities are defined by (11).

Equation 13 does not easily lend itself to integration in the case of time-dependent hit probabilities, even though firing times are limited to the negative exponential distribution. If hit probabilities are constant with respect to time, the integration is straightforward producing Ancker's result

$$P(A) = p_A\gamma_A \exp(-p_B\gamma_B b) \Bigg\{ \exp\left[-\left(p_A\gamma_A + p_B\gamma_B(1+m)\right)\left(\frac{a-b}{m}\right)\right]$$

$$\cdot \left[\frac{1}{p_A\gamma_A + p_B\gamma_B} - \frac{1}{p_A\gamma_A + p_B\gamma_B(1+m)}\right]$$

$$+ \frac{1}{p_A\gamma_A + p_B\gamma_B(1+m)} \Bigg\} .$$

When m [illegible] time, [illegible], P(A) becomes [illegible] the [illegible] duel with [illegible]

1.4.3 Time-Dependent Firing Rate

The duel with time-dependent kill probabilities can be considered a special case of a duel with the firing rates and kill probabilities all dependent on time. Let $r(t)t$ be the probability that exactly one round is fired in the interval $(t, t + \Delta t)$. If the time-dependent firing rate is used instead of the constant rate, (9) becomes

$$dG(x)/dx = -\gamma(x)p(x)G(x) ,$$

(10) becomes

$$G(t) = \exp\left[-\int_0^t \gamma(x)p(x)dx\right] , \qquad (14)$$

and (11) is now

$$h(t) = \gamma(t)p(t)\exp\left[-\int_0^t \gamma(x)p(x)dx\right] , \qquad (15)$$

i.e., the distribution of killing events in time is a non-stationary Poisson process as before. If we assume

$$\frac{p_A(t)\gamma_A(t)}{p_B(t)\gamma_B(t)} = k , \quad t \geq 0 ,$$

where k is a constant and

$$\int_0^t p_A(x)\gamma_A(x)dx \to \infty \text{ as } t \to \infty ,$$

then substituting (14) and (15) into (12) leads directly to

$$P(A) = k/(k + 1) .$$

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Chapter 2

SOME THOUGHTS ON ANALYSIS OF DIFFERENTIAL MODELS OF COMBAT

Robert Farrell

This chapter presents some brief thoughts on analysis of differential models of combat. The thoughts are sketchy but are deemed sufficiently useful "back-of-the-envelope" ideas to be documented.

2.1 *Force Ratios*

Consider the functions m(t), n(t) satisfying

$$\frac{dm}{dt} = -\beta(R_o - vt)n(t) \qquad (v > 0)$$

$$\frac{dn}{dt} = -\alpha(R_o - vt)m(t) \;,$$

where the notations are the same as that employed in Part C of the report. Transforming variables so that $m(R_o - vt) = m(t)$, $n(R_o - vt) = n(t)$, we have

$$\frac{d\hat{m}}{dr} = v^{-1}\beta(r)\hat{n}(r)$$

$$\frac{d\hat{n}}{dr} = v^{-1}\alpha(r)\hat{m}(r)$$

with initial values M,N at range $r = R_o$. Then we find that

$$\frac{d\rho}{dr} = v^{-1}\beta(r) - v^{-1}\alpha(r)\rho^2 \;,$$

where $\rho = \frac{\hat{m}}{\hat{n}}$ is the "force-ratio." This first-order differential equation in ρ is termed the fundamental equation.

Now there is a solution to this equation for any initial point (r,ρ) and any v. We know that the solution for (r,ρ_1,v) does not cross the solution for (r,ρ_2,v) if $\rho_1 \neq \rho_2$. This leads to two conclusions:

(1) For fixed assault speed v at r and forward (lessening r), increases in the initial value of v give increases throughout;

(2) Among solutions with step function v's, the highest one at one point is the highest at all prior points (if the solutions are limited to those existing on the whole range).

There is a maximum solution for step function v's with $0 < \varepsilon \leq v \leq V < \infty$. This solution is equivalent to a particular step function v, which we will call the ratio-optimal (in range) policy. This policy is:

$$v = \varepsilon \text{ if } \beta - \alpha\left(\frac{\bar{m}}{\bar{n}}\right)^2 < 0$$

$$v = V \text{ if } \beta - \alpha\left(\frac{\bar{m}}{\bar{n}}\right)^2 > 0 ,$$

where $\bar{m},\bar{n}$ are the values of m and n generated by the step function policy operating out to the present range.

The solution is generated in the following manner:

Step 1

(a) If $\beta - \alpha\left(\frac{m}{n}\right)^2 > 0$, solve the fundamental equation with initial point $(r, \frac{m}{n}, V)$ and take $r^{(1)}$ to be that point on the solution where $\beta - \alpha\left(\frac{\bar{m}}{\bar{n}}\right)^2 = 0$.

(b) If $\beta - \alpha\left(\frac{m}{n}\right)^2 < 0$, solve the fundamental equation with initial point $(r, \frac{m}{n}, \varepsilon)$ and take $r^{(1)}$ to be that point on the solution where $\beta - \alpha\left(\frac{\bar{m}}{\bar{n}}\right)^2 = 0$.

Step 2

Solve the fundamental equation with initial point

$\left\{r^{(1)}, \frac{\bar{m}(r^{(1)})}{\bar{n}(r^{(1)})}, v^*\right\}$, where v^* is ε or V, appropriately.

Step 3

Continue step-wise in this manner.

It is worth noting that the arguments of this analysis do not carry over to either force-ratio maximization in the time domain or force-difference maximization. In both these cases, the proposition on noncrossing solutions does not immediately hold, which invalidates the argument.

2.2 A Linear Model of Combat

Consider a military force consisting of an amount X_j of force units of type $j(j=1,\ldots,J)$. A pair of forces in combat may be considered as represented by a single vector X whose components are measures of force unit strength for various type units. (Units are not considered of identical type unless they have identical allegiances.)

A differential equation

$$D_t X = F_s(t,X)X ,$$

where F is a linear transformation, is called a differential model of combat. The parameter s is a scenario and environment index, and t is a time parameter.

If F_s does not depend on X, we have a linear differential equation (for our fixed scenario s). Under the usual regularity conditions, we know that the solutions to this differential equation are of the form

$$X_s(t) = Z_s(t)X_s(0) ,$$

where Z is a linear transformation. Thus, if the combat is taken to end at a fixed time, the final force vector is a linear function of the initial force vector, X(0). The model is clearly unrealistic if any components of the force vector become less than 0 during the combat. Accordingly, we will consider cases in which this does not occur.

Under appropriate convergence conditions, an explicit power series solution for Z in terms of F is obtained as follows. Let

$$Z_s(t) = Z_s^{(0)} + Z_s^{(1)}t + Z_s^{(2)}t^2 + \ldots$$

and

$$F_s(t) = F_s^{(1)} + F_s^{(2)}t + F_s^{(3)}t^2 + \ldots,$$

Then[1]

$$nZ_s^{(n)} = \sum_{i=1}^{n} F_s^{(i)} Z_s^{(n-i)} \quad .$$

The first few terms of $Z_s^{(n)}$ are, explicitly ,

$$Z_s^{(0)} = I$$

$$Z_s^{(1)} = F_s^{(1)}$$

$$Z_s^{(2)} = \frac{1}{2} F_s^{(2)} + \frac{1}{2} F_s^{(1)2}$$

$$Z_s^{(3)} = \frac{1}{3} F_s^{(3)} + \frac{1}{3} F_s^{(2)} F_s^{(1)} + \frac{1}{6} F_s^{(1)} F_s^{(2)} + \frac{1}{6} F_s^{(1)3}$$

[1]See Appendix F, 2.

$$Z_s^{(4)} = \frac{1}{4} F_s^{(4)} + \frac{1}{4} F_s^{(3)} F_s^{(1)} + \frac{1}{8} F_s^{(2)2} + \frac{1}{8} F_s^{(2)} F_s^{(1)2}$$
$$+ \frac{1}{12} F_s^{(1)} F_s^{(3)} + \frac{1}{12} F_s^{(1)} F_s^{(2)} F_s^{(1)} + \frac{1}{24} F_s^{(1)2} F_s^{(2)}$$
$$+ \frac{1}{24} F_s^{(1)4}$$
$$\cdot \cdot \cdot \cdot$$

It is worth noting that the noncommutation of $F_s^{(i)}$ and $F_s^{(j)}$ means that this function Z is not the same as the function

$$Y = \sum G^i/i!$$

with

$$G = \int_t F \ .$$

This function Y is the direct formal solution of the differential equation for Z in the real numbers.[1]

Term-by-term comparison for the first five terms gives

$$Y^{(0)} = Z_s^{(0)}$$

$$Y^{(1)} = Z_s^{(1)}$$

$$Y^{(2)} = Z_s^{(2)}$$

$$Y^{(3)} = Z_s^{(3)} + \frac{1}{6} (F_s^{(2)} F_s^{(1)} - F_s^{(1)} F_s^{(2)})$$

[1]See Appendix F, 2.

$$Y^{(4)} = Z^{(4)} + \frac{1}{12}\left(F_s^{(3)} F_s^{(1)} - F_s^{(1)} F_s^{(3)}\right)$$

$$+ \frac{1}{24}\left(F_s^{(2)} F_s^{(1)2} - F_s^{(1)2} F_s^{(2)}\right)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

The loss vector, $X(t) - X(0)$, is $(Z_s(t) - I)X(0)$, a linear function of the initial forces vector.

Appendix F, 2

RECURSIVE RELATION FOR THE POWER-SERIES SOLUTION OF Z

Robert Farrell

Let

$$D_t X(t) = F_s(t)X(t) \tag{1}$$

and

$$X(t) = Z_s(t)X(0) \cdot \tag{2}$$

Then, differentiating (2), we have

$$D_t X(t) = [D_t Z_s(t)]X(0),$$

and substituting in (1),

$$[D_t Z_s(t)]X(0) = [F_s(t)Z_s(t)]X(0)$$

or, since this holds for general X(0),

$$D_t Z_s(t) = F_s(t)Z_s(t) ,$$

a matrix differential equation. If we expand F and Z in series, we obtain the recursion relation given in the text of Section 2.2. The analogous real differential equation,

$$D_t y(t) = f(t)y(t) ,$$

has the known solution

$$y(t) = \exp\left(\int_t f(t)dt\right) .$$

This leads to, on reversing the analogy, the function

$$Y(t) = \sum G^i/i!$$

where $G = \int_t F$, as mentioned in the text.

Chapter 3

AMMUNITION REQUIREMENTS BASED ON DIFFERENTIAL MODELS OF COMBAT

Robert Gruhl and W. P. Cherry

This chapter presents initial analysis to obtain information about the ammunition requirements when the dynamics of combat are described the differential models documented in preceding parts of this report. Ammunition *requirements*, as contrasted with expenditures, are the total quantities of ammunition which a force type would need to engage in any particular battle without suffering a stockout. This information provides some guidelines regarding the amount of ammunition a force should be supplied with prior to an engagement. Ammunition *expenditures* are defined as the number of rounds actually lost to the logistic system during an engagement. In general, this includes not only those rounds actually fired but those destroyed by enemy fire, or otherwise rendered useless. This information is useful for costing in planning studies and when analysis of sequential battles is considered.

Principal interest in this chapter is the determination of ammunition requirements. Except where noted the analysis employs the following assumptions:

(1) Ammunition available for each force group is distributed equally among all the units of that group at the outset of the battle,

(2) Transfer of ammunition between units during combat is impossible, i.e., there is no redistribution either within or among force groups.

It is also assumed that the force is not resupplied during the engagement, although this can be added in a straightforward manner in follow-on analyses. Using assumption (1), it is sufficient to calculate the requirements for a single unit of a force group; by assumption (2), the overall requirements for a force group are based on the maximum requirements for any single unit in that group.

3.1 Homogeneous-Force Model: Constant Attrition Rates

The differential model considered here has the form

$$\frac{dn}{dt} = -\alpha m(t) \tag{1}$$

$$\frac{dm}{dt} = -\beta n(t) \ , \tag{2}$$

where

$n(t)$ = number of surviving Red forces at time t after the engagement begins,

$m(t)$ = number of surviving Blue forces at time t after the engagement begins,

$\alpha[\beta]$ = Blue[Red] weapon attrition rate (assumed constant).

This model requires that all units begin and end fire simultaneously, and that all units fire independently of each other and at the same rate. Thus, neglecting random fluctuations

for the moment, each unit consumes ammunition at the same rate and each survivor will have consumed the same amount of ammunition. Let this quantity be denoted by ε_m, where the subscript m denotes that ε is applicable to the Blue force. Likewise, the subscript n will refer to the Red forces. Then, if

M = initial number of Blue forces,

N = initial number of Red forces, and

Q = total ammunition requirements for the force,

it is evident that

$$Q_m = M\varepsilon_m \qquad (3)$$

$$Q_n = N\varepsilon_n \; . \qquad (4)$$

To predict ε in this simple case, one simply observes that

$$\varepsilon_m = b_m T \qquad (5)$$

$$\varepsilon_n = b_n T \, , \qquad (6)$$

where:

b = firing rate common to all units of a force, and

T = duration of the engagement

The battle duration T is determined, as in [C, 1.0], by solving (1) and (2) for the surviving number of forces and finding the time for a force to be annihilated. If $\alpha M^2 > \beta N^2$, Blue annihilates Red in

$$T_m = \frac{1}{\sqrt{\alpha\beta}} \tanh^{-1}\left[\sqrt{\beta/\alpha}\,\frac{N}{M}\right] . \tag{7}$$

Similarly, if $\alpha M^2 < \beta N^2$, Red annihilates Blue in

$$T_n = \frac{1}{\sqrt{\alpha\beta}} \tanh^{-1}\left[\sqrt{\alpha/\beta}\,\frac{M}{N}\right] . \tag{8}$$

Based on the developments of Part B, the attrition rate may be considered as

$$\alpha = \frac{1}{E[T]} = \frac{1}{c_1 + c_2 p} , \tag{9}$$

where

$E[T]$ = mean time to fire p rounds,

c_1, c_2 = acquisition and firing time constants of the blue weapon system (see equation 11 [B, 2.0]),

p = mean number of rounds required to destroy one enemy unit.

Thus, at any time, the rate of fire of a single Blue unit is

$$b_m = \alpha p_m , \tag{10}$$

where dimensionally,

b_m = rounds/unit time,

$$\alpha = \frac{\text{Red units killed}}{\text{single Blue unit} \times \text{unit time}} ,$$

$$p_m = \frac{\text{rounds fired}}{\text{Red units killed}} .$$

Thus,

$$b_m = \frac{p_m}{c_1 + c_2 p_m} . \tag{11}$$

Similarly,

$$b_n = \frac{p_n}{\bar{c}_1 + \bar{c}_2 p_n} . \tag{12}$$

Assume that initial ammunition supplies are q_m and q_n rounds per Blue and Red unit, respectively. We consider the case $\alpha M^2 > \beta N^2$ for which the outcome is annihilation of Red at $t = T_m$, provided there exists adequate ammunition on both sides. Thus, adequate supplies are[1]

$$q_m^* = b_m T_m$$

$$= \left(\frac{p_m}{c_1 + c_2 p_m}\right) \frac{1}{\sqrt{\alpha\beta}} \tanh^{-1} \left[\sqrt{\beta/\alpha}\, \frac{N}{M}\right] \tag{13}$$

and

$$q_n^* = b_n T_m$$

$$= \left(\frac{p_n}{\bar{c}_1 + \bar{c}_2 p_n}\right) \frac{1}{\sqrt{\alpha\beta}} \tanh^{-1} \left[\sqrt{\beta/\alpha}\, \frac{N}{M}\right] . \tag{14}$$

[1]The time constants for Red weapons are $\bar{c}_1$ and $\bar{c}_2$.

Results of battle are the same for supplies greater than q_m^* and q_n^*. Suppose, however, that lesser amounts q_m', q_n' are supplied. In this case battle ends because a force runs out of ammunition at T_s, where

$$T_s = \min\left\{ \frac{q_m'}{\dfrac{p_m}{c_1 + c_2 p_m}} , \quad \frac{q_n'}{\dfrac{p_n}{\bar{c}_1 + \bar{c}_2 p_n}} \right\} . \tag{15}$$

It is of interest to consider relaxing assumption (2) and assume instead that there exists a central supply of ammunition from which each unit draws. The rate of fire of the Blue *force* at time t is

$$b_M = \alpha p_m m(t) . \tag{16}$$

In a small interval of time, dt, the total ammunition expenditure is

$$e_M(t) = \alpha p_m m(t) dt , \tag{17}$$

and we obtain the total ammunition expenditure up to time t as

$$E_M(t) = \int_0^t \alpha p_m m(x) dx . \tag{18}$$

Substituting (9) and m(t) from equation 12 [C, 1.0], we obtain

$$E_M(t) = Np_m - p_m(N \cosh \sqrt{\alpha\beta}\, t) - \sqrt{\alpha/\beta}\, M \sinh \sqrt{\alpha\beta}\, t) . \tag{19}$$

Note that $E_m(t)$ is simply the product of the number of Red units destroyed at time t and the mean number of rounds required to destroy a single Red unit. Thus for the Red force ammunition expenditure at time t we obtain

$$E_N(t) = Mp_n - p_n(M \cosh \sqrt{\alpha\beta}\, t - \sqrt{\beta/\alpha}\, N \sinh \sqrt{\alpha\beta}\, t). \quad (20)$$

Again assuming $\alpha M^2 > \beta N^2$, we obtain for "adequate" ammunition supplies $Q_M^* = Np_m$ and $Q_N^* = Mp_n$ for the Blue and Red forces, respectively. Given supplies of $Q_M \leq Q_M^*$ and $Q_N \leq Q_N^*$, the duration of battle is now $T_s' = \min\{T_M', T_N'\}$, where T_M' is obtained from

$$Q_M = Np_m - p_m\left[N \cosh(\sqrt{\alpha\beta}\, T_M') - \sqrt{\alpha/\beta}\, M \sinh(\sqrt{\alpha\beta}\, T_M')\right]. \quad (21)$$

Substituting for the hyperbolic functions,

$$(\sqrt{\alpha/\beta}\, M - N)(e^{\sqrt{\alpha\beta}T_M'})^2 - 2\left(\frac{Q_M}{p_m} - N\right)e^{\sqrt{\alpha\beta}T_M'} - (\sqrt{\alpha/\beta}\, M + N) = 0, \quad (22)$$

which yields

$$T_M' = \frac{1}{\sqrt{\alpha\beta}} \ln\left[\frac{2(Q_M/p_m - N) \pm \sqrt{4(Q_M/p_m - N)^2 + 4(\alpha/\beta\, M^2 - N^2)}}{2(\sqrt{\alpha/\beta}\, M - N)}\right]. \quad (23)$$

Since in the cases of interest $Q_M \leq p_m N$, we need only consider the positive square root in (23).

Identical reasoning yields

$$T'_N = \frac{1}{\sqrt{\alpha\beta}} \ln \left[\frac{2(Q_N/p_n - M) + \sqrt{4(Q_N/p_n - M)^2 + 4\left(\frac{\beta}{\alpha} N^2 - M^2\right)}}{2\left(\frac{\beta}{\alpha} N - M\right)} \right]. \tag{24}$$

Under the assumption that $\alpha M^2 > \beta N^2$ the following battle outcomes result, dependent on initial ammunition supply:

Case 1: $Q_M \geq Q_M^*$ $\quad Q_N \geq Q_N^*$

Blue annihilates Red at

$$T_M = \frac{1}{\sqrt{\alpha\beta}} \tanh^{-1}\left[\sqrt{\beta/\alpha}\ \frac{N}{M}\right].$$

Case 2: $Q_M \geq Q_M^*$ $\quad Q_N < Q_N^*$

Red breaks off or surrenders to Blue because of ammunition stockout at $T'_N < T_n$.

Case 3: $Q_M < Q_M^*$ $\quad Q_N \geq Q_N^*$

Blue breaks off or surrenders to Red due to ammunition stockout at $T'_M < T_m$.

Case 4: $Q_M < Q_M^*$ $\quad Q_N < Q_N^*$

The battle ends at T'_s, where T'_s - min $\{T'_M, T'_N\}$, the time of first ammunition stockout.

The above formulation of supply from a central point is analogous to the somewhat unrealistic situation in which the ammunition of a destroyed unit is redistributed amongst the

surviving units, thus utilizing all ammunition. Since the derivations of $E_M(t)$ and $E_N(t)$ are based on the differential equations describing combat losses, the expenditure expressions hold for the results of the previous section where no redistribution was possible and each unit had an identical supply, q_m and q_n, respectively.

Thus for the Blue force at time t

(a) $$m(t) = M \cosh \sqrt{\alpha\beta}\, t - \sqrt{\beta/\alpha}\, N \sinh \sqrt{\alpha\beta}\, t \qquad (25)$$

(b) $$E_M(t) = Np_m - p_m(N \cosh \sqrt{\alpha\beta}\, t - \sqrt{\alpha/\beta}\, M \sinh \sqrt{\alpha\beta}\, t) \qquad (26)$$

(c) number of rounds available to survivors

$$\begin{aligned} n_A(t) &= (q_m - b_m t)m(t) \\ &= \left(q_m - \frac{p_m t}{c_1 + c_2 p_m}\right) m(t) \end{aligned} \qquad (27)$$

(d) number of rounds lost due to destroyed Blue units

$$\begin{aligned} n_L(t) &= q_m M - E_M(t) - n_A(t) \\ &= q_m(M - m(t)) - p_m(N - n(t)) + \frac{p_m t}{c_1 + c_2 p_m}\, m(t). \end{aligned} \qquad (28)$$

Similar results hold for the Red forces.

3.2 Homogeneous-Force Model: Range-Dependent Attrition Rates

We consider next homogeneous-force battles with attrition rates that are functions of the range, r. Hence,

$$\frac{dn}{dt} = -\alpha(r)m \tag{29}$$

$$\frac{dm}{dt} = -\beta(r)n \quad . \tag{30}$$

Letting P(r) represent the number of rounds required to destroy an enemy unit at range r, the firing rates may be estimated as

$$b_m(r) = \alpha(r)p_m(r) = \text{Blue firing rate at range } r \tag{31}$$

$$b_n(r) = \beta(r)p_n(r) = \text{Red firing rate at range } r. \tag{32}$$

Since the calculation of requirements depends upon how the engagement is conducted, we consider a constant speed attack engagement. In this case, one force attacks the stationary opposing force at a constant velocity, v. Let $\hat{r}$ represent the distance between the forces at the end of the battle. Then the number of rounds fired while at range r for a short interval dt is (for Blue)

$$b_m(r)dt = \frac{b_m(r)dr}{v} \quad . \tag{33}$$

Hence, the requirement is found from

$$\varepsilon_m = \int_{R_\alpha}^{\hat{r}} \frac{b_m(r)dr}{v} = \frac{1}{v}\int_{R_\alpha}^{\hat{r}} b_m(r)dr , \qquad (34)$$

where R_α (R_β) is the maximum range at which the Blue (Red) weapons can destroy targets. Analogously, the Red force requirement is

$$\varepsilon_n = \frac{1}{v}\int_{R_\beta}^{\hat{r}} b_n(r)dr \quad . \qquad (35)$$

As an example, assume that

$$b_m(r) = \begin{cases} K_m(R_\alpha - r) & 0 \le r \le R_\alpha \\ 0 & r > R \end{cases} \qquad (36)$$

$$b_n(r) = \begin{cases} K_n(R_\beta - r) & 0 \le r \le R_\beta \\ 0 & r > R_\beta , \end{cases} \qquad (37)$$

i.e., the firing rate is linear with respect to range and has slope K_m (K_n) for the Blue (Red) forces ($K_m, K_n < 0$) . Hence,

$$\varepsilon_m = \frac{1}{v}\int_{R_\alpha}^{\hat{r}} K_m(R_\alpha - r)\, dr$$

$$= \frac{K_m}{v}\left[R_\alpha \hat{r} - \frac{\hat{r}^2}{2} - \frac{R_\alpha^2}{2} \right] \quad . \qquad (38)$$

Similarly, for the Red forces, one has

$$\epsilon_n = \frac{K_n}{v}\left[R_\beta \hat{r} - \frac{\hat{r}^2}{2} - \frac{R_\beta^2}{2}\right] . \tag{39}$$

3.3 Heterogeneous-Force Model: Constant-Attrition Rates

We briefly consider the case of heterogeneous-force battles where the attrition rates of each of the weapon systems is constant. This situation is examined in detail in Part D of the report. The describing equations are given as

$$\frac{dm_i}{dt} = -\sum_{j=1}^{J} h_{ji}\beta_{ji}n_j \qquad i = 1,\ldots,I \tag{40}$$

$$\frac{dn_j}{dt} = -\sum_{i=1}^{I} e_{ij}\alpha_{ij}m_i \qquad j = 1,\ldots,J , \tag{41}$$

where:

m_i = number of surviving Blue forces of group i at time t,

n_j = number of surviving Red forces of group j at time t,

α_{ij} = constant rate at which a single Blue unit of group i can destroy Red group j units,

β_{ji} = constant rate at which a single Red unit of group j can destroy Blue group i units,

h_{ji} = fraction of the Red group j forces firing on the Blue group i forces,

e_{ij} = fraction of the Blue group i forces firing on the Red group j forces.

The reader is referred to [D, 1.0] for the solution to these coupled differential equations.

We define

t_{ij} = total length of time for which Blue group i fires on Red group j,

$\bar{t}_{ji}$ = total length of time for which Red group j fires on Blue group i,

which are determined from the optimal assignment strategies discussed in [D, 2.0]. In general,

$$\sum_{j=1}^{J} t_{ij} + t_i^* = T \qquad i = 1,\ldots,I \tag{42}$$

$$\sum_{i=1}^{I} \bar{t}_{ji} + \bar{t}_j^* = T \qquad j = 1,\ldots,J\ , \tag{43}$$

where

t_i^* = time between the annihilation of Blue force group i, if any, and the end of the engagement,

$\bar{t}_j^*$ = time between the annihilation of Red force group j, if any, and the end of the engagement,

T = duration of the engagement.

If

p_{ij} = number of rounds required for one weapon in Blue group i to destroy one target in Red group j, and

$\bar{p}_{ji}$ = number of rounds required for one weapon in Red group j to destroy one target in Blue group i,

and it is assumed that optimal strategies are always followed by both sides, then the ammunition requirements for Blue force group i are expressed by

$$\varepsilon_i = \sum_{j=1}^{J} \alpha_{ij} p_{ij} t_{ij} \qquad i = 1,\ldots,I \quad . \tag{44}$$

Similarly, for Red force group j, the requirements will be

$$\bar{\varepsilon}_j = \sum_{i=1}^{I} \beta_{ji} \bar{p}_{ji} \bar{t}_{ji} \qquad j = 1,\ldots,J \quad . \tag{45}$$

If there are M_i of the Blue group i units initially, and N_j of the Red group j units, then the total requirements for all force groups are

$$Q_i = M_i \varepsilon_i \qquad i = 1,\ldots,I \tag{46}$$

for the Blue forces, and

$$\bar{Q}_j = N_j \bar{\varepsilon}_j \qquad j = 1,\ldots,J \tag{47}$$

for the Red forces.

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13. ABSTRACT

This report describes research effort to develop analytical methods of defense processes, principally the combat process. The basic structure and underlying assumptions of generalized differential models of combat are presented. Research on methods to predict input attrition coefficients (comprised of attrition rates, allocation factors, and intelligence factors) is described. Attrition-rate models are presented for a spectrum of weapon system classes. Methods and associated research problems in solving the differential combat models for homogeneous- and heterogeneous-force battles are described. The effects of mobility in some simple engagements are analyzed. Optimal procedures for allocation of fire in constant attrition-coefficient, heterogeneous-force battles are developed. Preliminary modeling of the reconnaissance activity when line-of-sight is intermittent is presented. Areas for future research are described.

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